Probability:

Understanding the likelihood of something happening.

Properties of probabilities

They are numbers between 0 and 1

0 means the event under consideration is impossible

1 means the event under consideration is certain

Sometimes expressed as a percent or as a fraction

Not the same as odds

Very commonly used in real world applications:

OPENING PROBLEM

Life Insurance Companies use statistics on **life expectancy** and **death rates** to work out the premiums to charge people who insure with them.

The **life table** shown is from Australia. It shows the number of people out of 100 000 births who survive to different ages, and the expected years of remaining life at each age.

LIFE TABLE					
Male			Female		
Age	*Number surviving*	*Expected remaining life*	*Age*	*Number surviving*	*Expected remaining life*
0	100 000	73.03	0	100 000	79.46
5	98 809	68.90	5	99 307	75.15
10	98 698	63.97	10	99 125	70.22
15	98 555	59.06	15	98 956	65.27
20	98 052	54.35	20	98 758	60.40
25	97 325	49.74	25	98 516	55.54
30	96 688	45.05	30	98 278	50.67
35	96 080	40.32	35	98 002	45.80
40	95 366	35.60	40	97 615	40.97
45	94 323	30.95	45	96 997	36.22
50	92 709	26.45	50	95 945	31.59
55	89 891	22.20	55	94 285	27.10
60	85 198	18.27	60	91 774	22.76
65	78 123	14.69	65	87 923	18.64
70	67 798	11.52	70	81 924	14.81
75	53 942	8.82	75	72 656	11.36
80	37 532	6.56	80	58 966	8.38
85	20 998	4.79	85	40 842	5.97
90	8416	3.49	90	21 404	4.12
95	2098	2.68	95	7004	3.00
99	482	2.23	99	1953	2.36

For example, we can see that out of 100 000 births, 98 052 males are expected to survive to the age of 20, and from that age the survivors are expected to live a further 54.35 years.

Things to think about:

- Can you use the life table to estimate how many years you can expect to live?
- Can you estimate the probability of a new-born boy or girl reaching the age of 15?
- Can the table be used to estimate the probability that:
  - a 15 year old boy *will* reach age 75
  - a 15 year old girl *will not* reach age 75?
- An insurance company sells policies to people to insure them against death over a 30-year period. If the person dies during this period, the beneficiaries receive the agreed payout figure. Why are such policies cheaper to take out for a 20 year old than for a 50 year old?
- How many of your classmates would you expect to be alive and able to attend a 30 year class reunion?

Experimental probability:

The ***relative frequency*** of something occurring.
(Relative as compared to something else occurring.)

Find the experimental probability of the pig landing in various orientations.

Calculate $\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$

Landing orientation	Probability
Back	
Feet	
Side (dot up)	
Half Side (dot up)	
Side (dot down)	
Ear	
Snout	

Sample Space: The set of all possible outcomes of an experiment.

Often denoted as *U*. (as in ***U***niverse)

What is the sample space for pig landing orientations?

Representations of Sample Space:

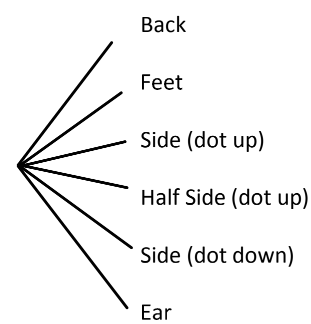
Lists

Standing
Back
Side (dot down)
Side (dot up)
Snout
Tailstand

Tables

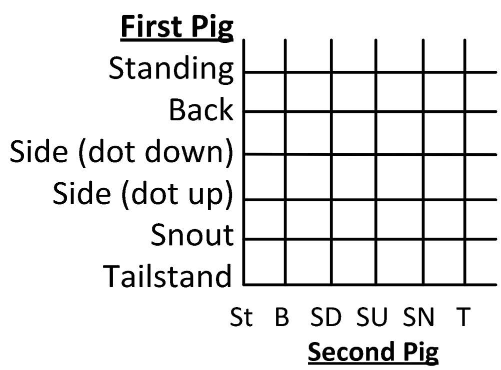
Landing orientation	Probability
Standing	
Back	
Side (dot down)	
Side (dot up)	
Snout	
Tailstand	
Other	

Tree Diagrams



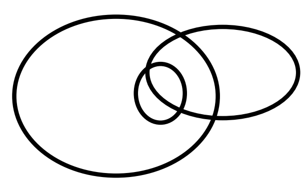
Grid Diagrams

(Experiments with 2 trials)



Venn Diagrams

(More on this later)



Theoretical Probability

In some situations symmetry will define an expected probability. Consider:

A fair die A fair coin A spinner with equal (unequal?) areas
A fair die with sides labelled 1,1,1,2,3,4
The sum of the values when two dice are rolled (Craps!)
The result of a Roulette spin

Among ***equally likely*** events, the mathematical or theoretical probability of an event *A* is given by:

$$P(A) = \frac{\text{the number of members of the event } A}{\text{the total number of possible outcomes}} = \frac{n(A)}{n(U)}.$$

Two events are ***complementary events*** if exactly one of them must occur.

Examples:
Heads & Tails
Roll a 6 and Roll something other than a 6 (Not 6)

The complement of *A* is written as *A*'.

Some properties of theoretical probability:

$0 \le P(A) \le 1$ By definition

$P(A) + P(A') = 1$ Something must occur!

HW:	**15A**	**Opening problem 1-4**
	15B	**#1-3**
	15C.1	**#1,3,5,6,7**
	15C.2	**#1-3**

Tables of outcomes

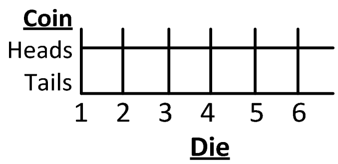
A helpful representation when there are two ***categorical*** variables to represent:

Hair Color

	Brown	Black	Blonde	Red
Men	15	10	4	1
Women	12	5	15	3

Compound Events

Exploring the probability of two or more events occurring.
Consider flipping a coin and rolling a die.
What is the probability of getting a heads and a 5?



Independent Compound Events

If the occurrence of event *A* does not affect the occurrence of event *B* (and vice versa) then *A* and *B* are called ***independent events***.

For independent events
P(*A* and *B*) = P(*A*)• P(*B*)

Similarly for more than two independent events:
P(*A* and *B* and *C*) = P(*A*)• P(*B*)• P(*C*)

Dependent Compound Events

If the occurrence of event *A* ***does*** affect the occurrence of event *B* (or vice versa) then *A* and *B* are called ***dependent events***.

Consider a bowl with 5 red marbles and 5 green marbles. What is the probability of selecting a red then a green marble?

P(*Red*) = 0.5 but then P(*Green*) = $\frac{5}{9}$

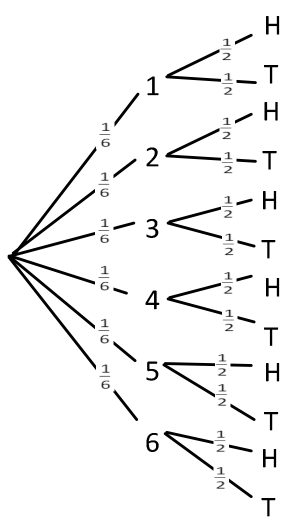
For dependent events
P(*A* ***then*** *B*) = P(*A*)• P(*B* given that *A* occurred)

HW:	**15D**	**#1-3**
	15E.1	**#1-6**
	15E.2	**#1-4**

Using Tree Diagrams

Very powerful - and will be even more useful as we proceed!

Consider rolling a die, then flipping a coin (independent events)



P(1 then H) = $\frac{1}{12}$

P(1 then T) = $\frac{1}{12}$

P(2 then H) = $\frac{1}{12}$

P(2 then T) = $\frac{1}{12}$

P(3 then H) = $\frac{1}{12}$

P(3 then T) = $\frac{1}{12}$

P(4 then H) = $\frac{1}{12}$

P(4 then T) = $\frac{1}{12}$

P(5 then H) = $\frac{1}{12}$

P(5 then T) = $\frac{1}{12}$

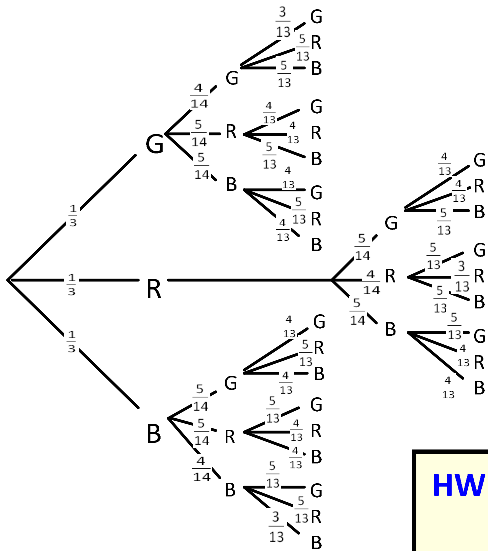
P(6 then H) = $\frac{1}{12}$

P(6 then T) = $\frac{1}{12}$

The probability of a particular sequence is the product along that path.

Tree Diagrams can also be used for dependent events.

Consider a box with 5 green, 5 red, and 5 blue balls.
You select three balls ***without replacement***.
Sampling without replacement creates ***dependent events***.
Probability of a particular sequence is the product along the paths



HW:	**15F**	**#1-6**
	15G	**#1-9**

Binomial Probabilities

A phenomenon with exactly two possible outcomes is called ***binomial***.

Binomial probabilities have special characteristics:

If P(A) = p, then P(A') = 1 - p	Since total probability must be 1.

A common question:
What is the probability of getting exactly x occurrences of A in n trials?

For example, I flip a coin 10 times. What is the probability of getting exactly 5 heads? Guesses?

Start with some smaller numbers to reveal a pattern. Consider 2 trials:

HH 1
HT
TH 2
TT 1

HHH 1
HHT
HTH 3
HTT
THH
THT 3
TTH
TTT 1

HHHH 1
HHHT
HHTH 4
HHTT
HTHH
HTHT
HTTH
HTTT
THHH
THHT 6
THTH
THTT
TTHH
TTHT 4
TTTH
TTTT 1

Pascal's Triangle!

n = 0	1						
n = 1	1	1					
n = 2	1	2	1				
n = 3	1	3	3	1			
n = 4	1	4	6	4	1		
n = 5	1	5	10	10	5	1	
n = 6	1	6	15	20	15	6	1
r =	0	1	2	3	4	5	6

The element in row n column r is given by:

$$C_r^n = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

These are called the ***binomial coefficients***
More easily found on your calculator!

So the number of ways to get 5 heads in 10 flips is:

$$C_5^{10} = {}_{10}C_5 = \binom{10}{5} = \frac{10!}{5!(10-5)!} = \frac{10\bullet9\bullet8\bullet7\bullet6\bullet5\bullet4\bullet3\bullet2}{5\bullet4\bullet3\bullet2(5\bullet4\bullet3\bullet2)}$$

$$= \frac{10\bullet9\bullet8\bullet7\bullet6}{5\bullet4\bullet3\bullet2} = \frac{9\bullet8\bullet7\bullet6}{4\bullet3} = 9\bullet2\bullet7\bullet2 = 36\bullet7 = 252$$

The total number of possible outcomes for n flips is the sum of the values in row n.

Notice that this is given by 2^n

Hence, the probability of getting exactly 5 heads in 10 flips is:

$$\frac{252}{2^{10}} = \frac{252}{1024} \approx 0.246 \text{ or } 24.6\%$$

In the example above (a fair coin) the probability of getting heads and tails is the same. More generally,

> Given a binomial phenomenon with P(A) = p and P(A') = 1 - p
> the probability that A occurs x times in n trials is given by
>
> $$\binom{n}{x} p^x (1-p)^{n-x}$$

On a TI-84, this is given by: **binompdf**(n, p, x).

For example, **binompdf**(*10*, *0.5*, *5*) = 0.246 as above.

Find the probability of getting exactly 3 ones in 10 rolls of a fair die.
Write out the calculation, find it manually using your calculator for computations then verify using **binompdf** on your calculator.

Other kinds of questions

a) Find the probability of getting ***no more than*** 3 ones in 10 rolls of a fair die.

No more than means rolling 0, 1, 2, or 3 ones. Add those probabilities together!
It's called the ***cumulative probability***.
Find it using your calculator with **binomcdf**(10, 1/6, 3) = 0.930

b) Find the probability of getting ***at least*** 3 ones in 10 rolls of a fair die.

At least 3 means rolling 3, 4, 5, or 6 ones. Add those probabilities together!
You can use **binomcdf** on your calculator but be careful:

> **binomcdf** always calculates the probability of getting ***no more than x***.

To find at least x, calculate **1 - binomcdf**(n, p, x - 1)....(can you explain why we use x - 1?)
In this case, **1 - binomcdf**(10, 1/6, 2)

HW:	**15H**	**#1-9**

1 SETS AND VENN DIAGRAMS

Venn Diagrams

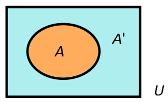
A way to represent the relationship between events.

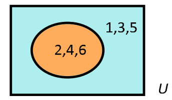
Sample space *U* is drawn as a rectangle (or the outermost oval)

Events of interest are drawn with circles and ovals containing the favorable outcomes of that event.

Overlapping regions indicate that an outcome is part of multiple events.

Areas are ***not*** drawn proportionally to the number of outcomes inside



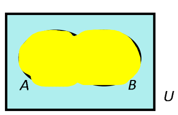


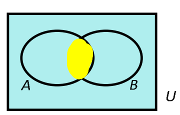
Getting an even roll of a die

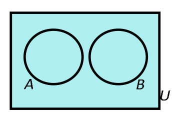
Set Notation

The following notation is taken from the HL IB notation sheet

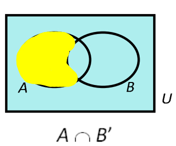
$\{x_1, x_2, \ldots\}$	the set with elements $x_1, x_2, \ldots$	$A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$
$n(A)$	the number of elements in the finite set A	$n(A) = 3$
$\{x \mid \quad\}$	the set of all x such that	$A = \{x \mid x \in \{1,2,3\}\}$
$\in$	is an element of	$2 \in \{1, 2, 3\}$ or $2 \in A$
$\notin$	is not an element of	$5 \notin \{1, 2, 3\}$ or $5 \notin A$
$\varnothing$	the empty (null) set	$\{x \mid x + 2 = x + 7\} = \varnothing$
U	the universal set	Context specific: coin flip $U = \{H, T\}$
$\cup$	union	$A \cup B = \{1, 2, 3, 4, 5\}$ (aka logical ***and***)
$\cap$	intersection	$A \cap B = \{3\}$ (aka logical ***or***)
$\subset$	is a proper subset of	$\{1, 2\} \subset A$
$\subseteq$	is a subset of	$\{1, 2, 3\} \subseteq A$ but $\{1, 2, 3\} \not\subset A$
A'	the complement of the set A	$A' = \{x \mid x \in U \cap x \notin A\}$ or $A' = \{x \mid x \in \mathbb{R} \cap x \notin \{1, 2, 3\}\}$
$A \times B$	the Cartesian product of sets A and B (that is, $A \times B = \{(a, b) \mid a \in A,\ b \in B\}$)	$A \times B = \{(1,3),(1,4),(1,5), (2,3),(2,4),(2,5),(3,3),(3,4),(3,5)\}$



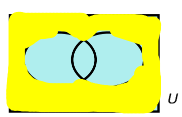




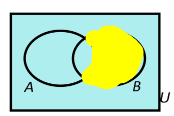
Disjoint sets
(Mutually exclusive)



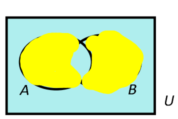
$A \cap B'$



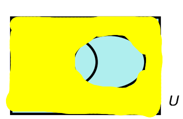
$A' \cap B' = (A \cup B)'$



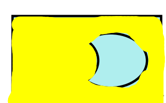
$A' \cap B$



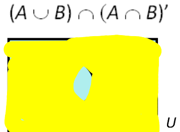
$(A \cup B) \cap (A \cap B)'$



B'



$A \cup B'$



$A' \cup B' = (A \cap B)'$

Conjecture: Set operations are commutative but not associative.
Can you prove this?

Some Set Identities

$$P(A' \cup B') = P((A \cap B)') = 1 - P(A \cap B)$$

$$P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$$

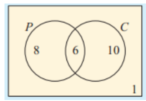
J LAWS OF PROBABILITY

Consider:

In a class of 25 students, 14 like pizza and 16 like iced coffee. One student likes neither and 6 students like both. One student is randomly selected from the class.

What is the probability that the student likes either pizza or coffee?

Draw a Venn Diagram



Notice that when we add $n(P) + n(C)$ (14 + 16) we are counting 6 students twice.

To find our answer, we can subtract: $n(P) + n(C) - n(P \cap C)$

In general, then the ***Addition Law of Probability*** states that:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

...or in words...

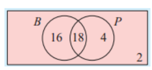
$$P(\textbf{either } A \textbf{ or } B) = P(A) + P(B) - P(\textbf{both } A \textbf{ and } B).$$

Note that for ***disjoint events***, P(both A and B) = 0. So for this case

$$P(A \cup B) = P(A) + P(B).$$

Is the converse true? Can you prove it?

Try one:
In a class of 40 students, 34 like bananas, 22 like pineapples, and 2 dislike both. What is the probability that a student likes either bananas or pineapples?



$$P(B \cup P) = P(B) + P(P) - P(B \cap P)$$

$$P(B \cup P) = \frac{34}{40} + \frac{22}{40} - \frac{18}{40} = \frac{38}{40} = \frac{19}{20}$$

Conditional Probability

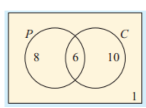
A* | *B means that A occurs, given that B has already occurred.

Also written as ***A* given *B***.

In a class of 25 students, 14 like pizza and 16 like iced coffee. One student likes neither and 6 students like both. One student is randomly selected from the class.

What is the probability that a student who likes pizza also likes coffee?

Refer to the Venn Diagram



If we know they liked pizza, then there are only 14 people in the sample space.

Of those, 6 like coffee. So the answer is $\frac{6}{14}$ or $\frac{3}{7}$

Algebraically:

$$P(C \mid P) = \frac{P(C \cap P)}{P(P)}$$

Probability of liking both

Sample space is reduced because we know they like pizza

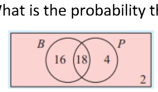
Probability of liking coffee given liking pizza

Generally the ***Conditional Law of Probability*** states that:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Notice that the position in the fraction ***is very important***! Understand why!

<u>**Your turn:**</u>
In a class of 40 students, 34 like bananas, 22 like pineapples, and 2 dislike both. What is the probability that a student who likes pineapples also likes bananas?



$$P(B \mid P) = \frac{P(B \cap P)}{P(P)} = \frac{\frac{18}{40}}{\frac{22}{40}} = \frac{18}{40} \cdot \frac{40}{22} = \frac{18}{22} = \frac{9}{11}$$

HW: 15I.1 #1-9
15I.2 #1-3
15J #1-11
Ready for Practice: #4,7,9,10,11,12,14

K INDEPENDENT EVENTS

Our previous definition
If the occurrence of event *A* does not affect the occurrence of event *B* (and vice versa) then *A* and *B* are called ***independent events***.

For independent events
$P(A \text{ and } B) = P(A) \cdot P(B)$

Using algebraic notation

$$A \text{ and } B \text{ are } \textbf{independent events} \quad \Leftrightarrow \quad P(A \cap B) = P(A)\,P(B).$$

Note the if and only if. This allows us to answer questions like:

A is getting heads on a coin, *B* is getting less than 3 on a die. ***Show that*** *A* and *B* are independent.

$$P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{3} \text{ so } P(A)\cdot P(B) = \frac{1}{6}$$

Using a grid or tree diagram, we can show that $P(A \text{ and } B) = P(A \cap B)$ is also $\frac{1}{6}$

Therefore, *A* and *B* are independent.

Another observation:
Refer back to the addition law of probability:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

But when *A* and *B* are independent, the last term is just P(*A*)P(*B*). So the implication is:

$$A \text{ and } B \text{ are independent} \Leftrightarrow P(A \cup B) = P(A) + P(B) - P(A)\cdot P(B)$$

Some problems that these ideas illuminate:

Example 27

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = p$. Find p if:

a A and B are mutually exclusive **b** A and B are independent.

a If A and B are mutually exclusive, $A \cap B = \varnothing$ and so $P(A \cap B) = 0$

But $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore \; p = \frac{1}{2} + \frac{1}{3} - 0 = \frac{5}{6}$

b If A and B are independent, $P(A \cap B) = P(A)\,P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$\therefore \; P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$ and hence $p = \frac{2}{3}$

Example 28

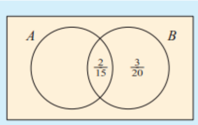
Given $P(A) = \frac{2}{5}$, $P(B \mid A) = \frac{1}{3}$ and $P(B \mid A') = \frac{1}{4}$ find:

a $P(B)$ **b** $P(A \cap B')$

$P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$ so $P(B \cap A) = P(B \mid A)\,P(A) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$

Similarly, $P(B \cap A') = P(B \mid A')\,P(A') = \frac{1}{4} \times \frac{3}{5} = \frac{3}{20}$

$\therefore$ the Venn diagram is:



a $P(B) = \frac{2}{15} + \frac{3}{20} = \frac{17}{60}$

b $P(A \cap B') = P(A) - P(A \cap B)$
$= \frac{2}{5} - \frac{2}{15}$
$= \frac{4}{15}$

HW: 15K #1-8
Practice: #13,15 (all)