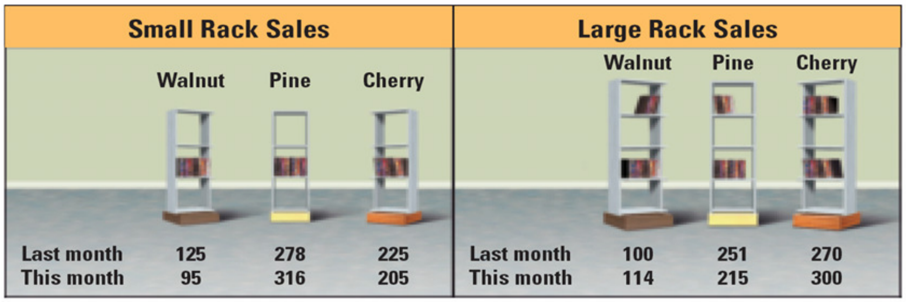
Matrix Operations

A company manufactures small and large steel DVD racks with wooden bases. Each size of rack is available in three types of wood: walnut, pine, and cherry. Sales of the racks for last month and this month are shown below.



We can organize the information into tables (in several ways):

Last Month

	Small	Large
Walnut	125	100
Pine	278	251
Cherry	225	270

This Month

	Small	Large
Walnut	95	114
Pine	316	215
Cherry	205	300

Small Racks

	Last Month	This Month
Walnut	125	95
Pine	278	316
Cherry	225	205

Large Racks

	Last Month	This Month
Walnut	100	114
Pine	251	215
Cherry	270	300

A table can be written as a "**matrix**" consisting of just the numbers.

$$\begin{pmatrix} 95 & 114 \\ 316 & 215 \\ 205 & 300 \end{pmatrix} \qquad \begin{pmatrix} 125 & 100 \\ 278 & 251 \\ 225 & 270 \end{pmatrix}$$

Matrix Vocabulary

Rows
Columns
Elements: a_{ij} is the element in the i^{th} row and j^{th} column
Matrices: the plural of matrix
Row Dimension: the number of rows in a matrix
Column Dimension: c'mon guys, whaddya think?
Dimensions of a Matrix: Convention is to give the ***rows first***!
Order of a matrix: The product of the # of rows and columns

$$\begin{pmatrix} 95 & 114 \\ 316 & 215 \\ 205 & 300 \end{pmatrix} \qquad \begin{pmatrix} 125 & 100 \\ 278 & 251 \\ 225 & 270 \end{pmatrix} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

This Month (A) Last Month (B) General Matrix

It's not hard, but you have to pay attention!

Questions:
What would you do to find the sales of both months combined?
What would you do to find the change in sales from last month to this month?
What would you do to find the average monthly sales for the two months?
What would you do to find the total annual sales if this month's sales were repeated every month for 12 months?
What would you do to predict sales 6 months from now if the change in sales from last month to this month continues?

What would you do to find the ***total sales*** of both months combined?

$$\begin{pmatrix} 95 & 114 \\ 316 & 215 \\ 205 & 300 \end{pmatrix} + \begin{pmatrix} 125 & 100 \\ 278 & 251 \\ 225 & 270 \end{pmatrix} = \begin{pmatrix} 220 & 214 \\ 594 & 466 \\ 430 & 570 \end{pmatrix}$$

This Month (*A*) Last Month (*B*) Total (*A* + *B* = *C*)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{pmatrix}$$

General Addition

A + ***B*** = ***C*** means $c_{ij} = a_{ij} + b_{ij}$

Question: Can you add any two matrices together?

Answer: NO! To add matrices, their dimensions must be the same!

What would you do to find the ***change in sales*** from last month to this month?

$$\begin{pmatrix} 95 & 114 \\ 316 & 215 \\ 205 & 300 \end{pmatrix} - \begin{pmatrix} 125 & 100 \\ 278 & 251 \\ 225 & 270 \end{pmatrix} = \begin{pmatrix} -30 & 14 \\ 38 & -36 \\ -20 & 30 \end{pmatrix}$$

Notice the order of subtraction!

This Month (*A*) Last Month (*B*) Change (*A* - *B*= *C*)

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \\ a_{31}-b_{31} & a_{32}-b_{32} \end{pmatrix}$$

General Subtraction

A - ***B*** = ***C*** means $c_{ij} = a_{ij} - b_{ij}$

Question: Can you subtract any two matrices?

Answer: NO! To subtract matrices, their dimensions must be the same!

What would you do to find the ***average monthly sales***?

First find the total:
$$\underset{\text{This Month } (A)}{\begin{pmatrix} 95 & 114 \\ 316 & 215 \\ 205 & 300 \end{pmatrix}} + \underset{\text{Last Month } (B)}{\begin{pmatrix} 125 & 100 \\ 278 & 251 \\ 225 & 270 \end{pmatrix}} = \underset{\text{Total } (A + B = C)}{\begin{pmatrix} 220 & 214 \\ 594 & 466 \\ 430 & 570 \end{pmatrix}}$$

Then divide by 2
(times .5)
$$\frac{1}{2}\underset{\boldsymbol{A} + \boldsymbol{B}}{\begin{pmatrix} 220 & 214 \\ 594 & 466 \\ 430 & 570 \end{pmatrix}} = \underset{\text{Average} = (\boldsymbol{A} + \boldsymbol{B})/2}{\begin{pmatrix} 110 & 107 \\ 297 & 233 \\ 215 & 285 \end{pmatrix}}$$

General multiplication by a ***scalar***
$k\boldsymbol{A} = \boldsymbol{B}$ means $b_{ij} = ka_{ij}$

$$k\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \\ ka_{31} & ka_{32} \end{pmatrix}$$

Question: Can you multiply any matrix by any scalar?

Answer: YES!

What would you do to find the total annual sales if this month's sales were repeated every month for 12 months?
What would you do to predict sales 6 months from now if the change in sales from last month to this month continues?

12***A*** = Annual sales if this month's sales continue.

A + 6(***A*** - ***B***) = Prediction for 6 months from now if changes continue

You can use matrices to represent multiple equations:

$$4\underbrace{\begin{pmatrix} 3x & 2 \\ 8 & w \\ 6 & 3 \end{pmatrix}}_{\boldsymbol{A}} + \underbrace{\begin{pmatrix} 5 & 7 \\ 3y & -2 \\ 1 & 2t \end{pmatrix}}_{\boldsymbol{B}} = \underbrace{\begin{pmatrix} -7 & z \\ 16 & 8 \\ 3m & 15 \end{pmatrix}}_{\boldsymbol{C}}$$

Can be written as:
4***A*** + ***B*** = ***C***

More importantly, we can manipulate matrices as symbols, rather than working with individual elements.

For example, 4(***A*** - ***B***) + 2***A*** = 4***A*** - 4***B*** + 2***A*** = 6***A*** - 4***B***
More on this later.

Multiplying matrices

Now suppose that small walnut cabinets sell for \$150, small Pine cabinets sell for \$100, and small Cherry cabinets sell for \$200 each.
We can organize the price information in a table as follows:

	Walnut	Pine	Cherry
Price	150	100	200

or as a matrix: $\begin{bmatrix} 150 & 100 & 200 \end{bmatrix}$

What calculation would you do to find the total income from small racks this month?

$\begin{bmatrix} 150 & 100 & 200 \end{bmatrix}$ is called a ***row vector***. (A matrix with a single row is called a ***vector***)

Let's organize the small rack sales for this month like this: $\begin{pmatrix} 95 \\ 316 \\ 205 \end{pmatrix}$
It's called a ...you guessed it ***column vector***.

Total income for small sales this month is the ***dot product*** of the row and column vectors.

$$\begin{bmatrix} 150 & 100 & 200 \end{bmatrix}\begin{pmatrix} 95 \\ 316 \\ 205 \end{pmatrix} = 150\cdot 95 + 100\cdot 316 + 200\cdot 205 = 86850$$

We can organize the total sales calculations for both months as follows:

$$\begin{array}{c} \textbf{\textit{Prices}} \\ \begin{array}{ccc}\text{Walnut} & \text{Pine} & \text{Cherry}\end{array} \\ \begin{bmatrix}150 & 100 & 200\end{bmatrix}\end{array} \begin{array}{cc} \textbf{\textit{Months}} & \\ \begin{array}{cc}\text{This} & \text{Last}\end{array} & \textbf{\textit{Sales}} \\ \begin{pmatrix}95 & 125\\ 316 & 275\\ 205 & 225\end{pmatrix} & \begin{array}{l}\text{Walnut}\\ \text{Pine}\\ \text{Cherry}\end{array}\end{array} \quad \begin{array}{c}\textbf{\textit{Income}} \\ \begin{array}{cc}\text{This} & \text{Last}\end{array} \\ =\begin{bmatrix}86850 & 91550\end{bmatrix}\end{array}$$

Row 1 times Col 1 = (150 * 95) + (100 * 316) + (200 * 205) = 86850 = Row 1 Col 1 or Element$_{1,1}$
Row 1 times Col 2 = (150 * 125) + (100 * 278) + (200 * 225) = 91550= Row 1 Col 2 or Element$_{1,2}$

To multiply two matrices, find the dot products of all the combinations of row and column vectors.

Multiplying Matrices
Calculate the dot product
of the ***i***th ***row*** of the first matrix
with the ***j***th ***column*** of the second matrix
to get the ***ij***th **element** of the resulting product

$$\underset{A}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} \bullet \underset{B}{\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}} = \underset{C}{\begin{pmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) \end{pmatrix}}$$

Try a few

$$\begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \bullet \begin{pmatrix} 2 & 6 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -7 & -6 \\ 19 & 32 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 7 \\ 0 & 9 \\ 2 & 1 \end{pmatrix} \bullet \begin{pmatrix} 5 & 2 \\ 4 & 6 \\ 0 & 8 \end{pmatrix} = \text{Goodluck Jim!}$$

$$\begin{pmatrix} 2 & 6 \\ 3 & 4 \end{pmatrix} \bullet \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 24 \\ 11 & 11 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 7 & 4 \\ 0 & 9 & -1 \\ 2 & 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} 5 & 2 \\ 4 & 6 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 13 & 68 \\ 36 & 46 \\ 14 & 10 \end{pmatrix}$$

Homework due Wednesday 11/30:
Ex 3: 13.2,13.3,13.4,14.2,14.3,14.8,15.5,15.9
H&H:11B.7 p. 288 #1-7

Review basic matrix operations in H&H Section 11.B pp. 276-291
Some highlights:

The Zero Matrix, often denoted as ***O*** is just a matrix of all zeroes.

The Opposite of a matrix is -***A*** or (-1)***A***. Each element of ***A*** is negated.

So ... ***A*** + (-***A***) = ***O***.

Can any two matrices be multiplied? What must be true?

The column dimension of the first matrix must = the row dimension of the second.

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \end{array} \bullet \begin{array}{cc} & p \\ n & \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix} \end{array} = \begin{array}{cc} & p \\ m & \begin{pmatrix} c_{11} & \cdots & c_{1p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{pmatrix} \end{array}$$

Matrix Algebra

Matrices can be manipulated using algebra, as long as you obey the rules we have discovered. The most important is that

Matrix Multiplication is not commutative!

Some examples:

Simplify $\boldsymbol{A}(\boldsymbol{B} - \boldsymbol{C})^2$

$= \boldsymbol{A}(\boldsymbol{B}^2 - 2\boldsymbol{BC} + \boldsymbol{C}^2) = \boldsymbol{AB}^2 - 2\boldsymbol{ABC} + \boldsymbol{AC}^2$

Given $\boldsymbol{AX} + \boldsymbol{B} = \boldsymbol{C}$, solve for $\boldsymbol{X}$

$\boldsymbol{AX} = \boldsymbol{C} - \boldsymbol{B}$	get variable term alone on one side
$\boldsymbol{A}^{-1}\boldsymbol{AX} = \boldsymbol{A}^{-1}(\boldsymbol{C} - \boldsymbol{B})$	***pre***multiply by $\boldsymbol{A}^{-1}$
$\boldsymbol{X} = \boldsymbol{A}^{-1}(\boldsymbol{C} - \boldsymbol{B})$	From definition of inverse.

If $\boldsymbol{A}^2 = 2\boldsymbol{A} + 3\boldsymbol{I}$, find an expression for $\boldsymbol{A}^{-1}$ in terms of $\boldsymbol{A}$ and $\boldsymbol{I}$

$\boldsymbol{A}^{-1}\boldsymbol{A}^2 = \boldsymbol{A}^{-1}(2\boldsymbol{A} + 3\boldsymbol{I})$	***pre***multiply by $\boldsymbol{A}^{-1}$
$\boldsymbol{A}^{-1}\boldsymbol{AA} = 2\boldsymbol{A}^{-1}\boldsymbol{A} + 3\boldsymbol{A}^{-1}\boldsymbol{I}$	Rewrite $\boldsymbol{A}^2$ and simplify RHS
$\boldsymbol{IA} = 2\boldsymbol{I} + 3\boldsymbol{A}^{-1}$	Simplify both sides
$\boldsymbol{A} - 2\boldsymbol{I} = 3\boldsymbol{A}^{-1}$	Solve for $\boldsymbol{A}^{-1}$
$\frac{1}{3}(\boldsymbol{A} - 2\boldsymbol{I}) = \boldsymbol{A}^{-1}$	Solve for $\boldsymbol{A}^{-1}$

Determinants of 2x2 Matrices

A ***square*** matrix has the same number of rows and columns.

The ***determinant*** of a square matrix is a particular number that describes the matrix:

It ***determines*** whether the matrix has an inverse (which we will discuss soon!)

Calculating the determinant of a 2x2 matrix:

Example: $\begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix}$ or $\det\begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix}$ = (2)(7) - (4)(3) = 14 - 12 = 2

In general: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = *ad* - *bc*

Show that for 2x2 matrices in general, $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$

Remember that a system of linear equations can be represented as a single matrix equation.

$$\underset{\boldsymbol{A}}{\begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}}\underset{\boldsymbol{x}}{\begin{pmatrix} x \\ y \end{pmatrix}}=\underset{\boldsymbol{b}}{\begin{pmatrix} 12 \\ 10 \end{pmatrix}} \text{ means } \begin{matrix} 3x+4y=12 \\ 2x-5y=10 \end{matrix} \text{ and can be written: } \boldsymbol{Ax} = \boldsymbol{b}$$

Like a regular equation, if we can "divide" by ***A*** we can find ***x***. But how do we "divide" by a matrix?

Think back to a scalar equation. $3x = 12$.
The next step can be divide both sides by 3 ... or ... multiply both sides by $\frac{1}{3}$

Notice that the result is that we get ***1*** times *x* because $\frac{1}{3}$ is the ***inverse*** of 3.

What do we multiply ***A*** by to get the matrix equivalent of **1**?
Well first, what is the matrix equivalent of 1?

What happens when you multiply $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 7 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$

How about $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix}$

An ***identity matrix*** is a *square* matrix that when multiplied by a matrix ***A*** gives ***A***.

That is: ***IA*** = ***A*** and also ***AI*** = ***A***

An identity matrix consists of all zeroes with ones along the diagonal.

3x3 identity matrix = $\boldsymbol{I}_3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

n x *n* identity matrix = $\boldsymbol{I}_n$ $\begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$ *n* rows

n cols

The ***inverse of a matrix A*** is the matrix that multiplies ***A*** to give the identity matrix.

So $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$ and also $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$

How do we calculate the inverse of a matrix?

The inverse of $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by:

$$A^{-1}=\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Example: find the inverse of $\begin{pmatrix} -4 & -6 \\ 4 & 7 \end{pmatrix}$

$$\begin{pmatrix} -4 & -6 \\ 4 & 7 \end{pmatrix}^{-1}=\frac{1}{\begin{vmatrix} -4 & -6 \\ 4 & 7 \end{vmatrix}}\bullet\begin{pmatrix} 7 & 6 \\ -4 & -4 \end{pmatrix}=\frac{1}{-28-(-24)}\bullet\begin{pmatrix} 7 & 6 \\ -4 & -4 \end{pmatrix}$$

$$=\frac{1}{-4}\bullet\begin{pmatrix} 7 & 6 \\ -4 & -4 \end{pmatrix}=\begin{pmatrix} \frac{-7}{4} & \frac{-3}{2} \\ 1 & 1 \end{pmatrix}$$

Find the inverse:

$$\begin{pmatrix} 2 & -7 \\ -1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$$

Solve the matrix equation:

$$\begin{pmatrix} 8 & 3 \\ 2 & 1 \end{pmatrix}\boldsymbol{x}=\begin{pmatrix} 5 & 4 \\ 2 & -3 \end{pmatrix} \qquad \begin{pmatrix} 6 & 8 \\ 2 & 3 \end{pmatrix}\boldsymbol{x}=\begin{pmatrix} 4 & 3 \\ 0 & -2 \end{pmatrix}$$

Solve the system of equations:

$2x - 4y = 16$
$3x + 5y = 24$

$$\begin{pmatrix} 2 & -4 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 16 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 & -4 \\ 3 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 16 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{\begin{vmatrix} 2 & -4 \\ 3 & 5 \end{vmatrix}}\begin{pmatrix} 5 & 4 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 16 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{10-(-12)}\begin{pmatrix} 5 & 4 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 16 \\ 24 \end{pmatrix}=\frac{1}{22}\begin{pmatrix} 5 & 4 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 16 \\ 24 \end{pmatrix}$$

$$=\frac{1}{22}\begin{pmatrix} 80+96 \\ -48+48 \end{pmatrix}=\begin{pmatrix} \frac{176}{22} \\ 0 \end{pmatrix}=\begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$17x - 13y = 95$
$13x + 11y = 19$

$$\begin{pmatrix} 17 & -13 \\ 13 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 95 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 17 & -13 \\ 13 & 11 \end{pmatrix}^{-1}\begin{pmatrix} 95 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{\begin{vmatrix} 17 & -13 \\ 13 & 11 \end{vmatrix}}\begin{pmatrix} 11 & 13 \\ -13 & 17 \end{pmatrix}\begin{pmatrix} 95 \\ 19 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{178-(-169)}\begin{pmatrix} 11 & 13 \\ -13 & 17 \end{pmatrix}\begin{pmatrix} 95 \\ 19 \end{pmatrix}=\frac{1}{347}\begin{pmatrix} 11 & 13 \\ -13 & 17 \end{pmatrix}\begin{pmatrix} 95 \\ 19 \end{pmatrix}$$

$$=\frac{1}{347}\begin{pmatrix} 1045+247 \\ -247+323 \end{pmatrix}=\begin{pmatrix} \frac{1292}{347} \\ \frac{76}{347} \end{pmatrix}$$

Wow - how about a calculator!

Calculating the determinant of a 3x3 matrix:

In General: $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a[ei - fh]$ $\qquad \begin{vmatrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 6 & 9 \end{vmatrix} = 2[(7)(9)-(4)(6)] = 2[39] = 78$

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - b[di - fg]$ $\qquad \begin{vmatrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 6 & 9 \end{vmatrix} - 3[(1)(9) - (4)(8)] = -3[-23] = +69$

$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + c[dh - eg]$ $\qquad \begin{vmatrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 6 & 9 \end{vmatrix} + 5[(1)(6) - (7)(8)] = 5[-50] = -250$

$78 + 69 - 250 = -103$

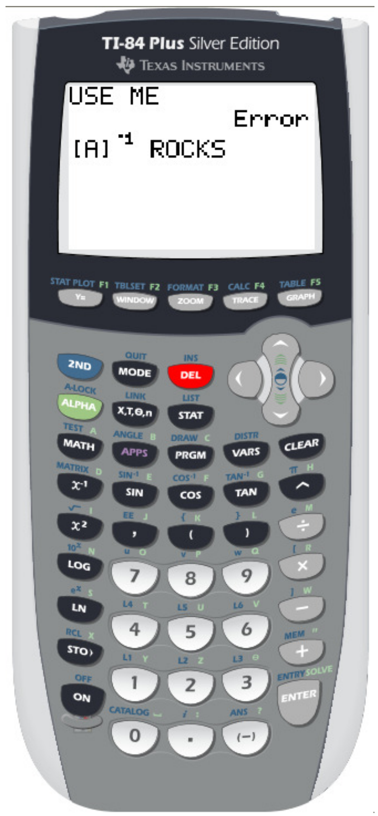
...or...

$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$ $\quad (aei + bfg + cdh) - (gec + hfa + idb)$

$\left|\begin{matrix} 2 & 3 & 5 \\ 1 & 7 & 4 \\ 8 & 6 & 9 \end{matrix}\right| \begin{matrix} 2 & 3 \\ 1 & 7 \\ 8 & 6 \end{matrix} = (2)(7)(9) + (3)(4)(8) + (5)(1)(6)$

$- [(8)(7)(5) + (6)(4)(2) + (9)(1)(3)] = -103$

How to calculate the inverse of a 3x3 matrix:



Find the inverse of: $\begin{pmatrix} 2 & -4 & 3 \\ -4 & 8 & -6 \\ 1 & 5 & 7 \end{pmatrix}$

Find the inverse of: $\begin{pmatrix} 3 & -4 & 15 \\ -4 & 8 & -20 \\ 1 & 7 & 5 \end{pmatrix}$

Try solving:

$x + y + z = 3$

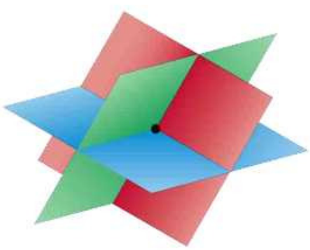
$4x + 4y + 4z = 7$

$3x - y + 2z = 5$

What happens? What does this represent? What does one equation represent?

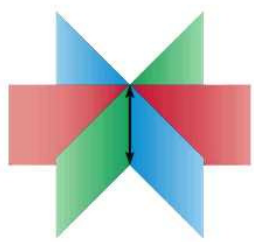
Exactly one solution

The planes intersect in a single point.



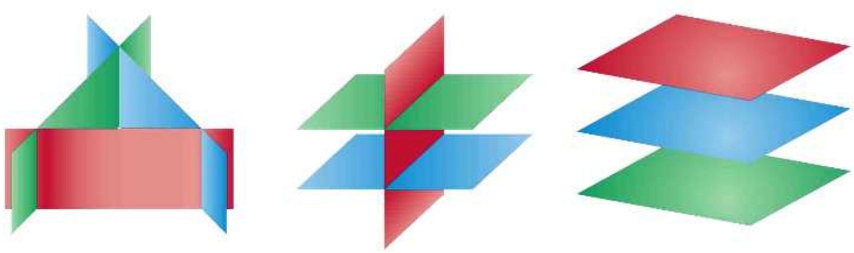
Infinitely many solutions

The planes intersect in a line or are the same plane.



No solution

The planes have no common point of intersection.



Some special matrices:

Definitions: HH 275
"Order" of a matrix = m x n
Zero matrix HH281
Negative HH281
Definition of equality HH 276
Addition & Subtraction HH 276
Scalar multiply HH279
Matrix algebra HH 281 - notation, more on 289
Multiplying matrices HH282
Identity matrices HH289
Determinant of a square matrix (2x2) HH 292
Find Inverse of a 2x2 HH292
Determinant of a square matrix (3x3) HH 297
Conditions for the existence of an inverse (det <> 0)
Connection between cross product and 3x3 determinant -HL Ext
Higher order determinants on calc - ext
Matrices in linear systems HH299

Cramer's rule $$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \ldots, n$$

where A_i is the matrix formed by replacing the ith column of A by the column vector b.

What good are determinants?
Take a look at what this means:

$$\underset{\boldsymbol{A}}{\begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}} \underset{\boldsymbol{x}}{\begin{pmatrix} x \\ y \end{pmatrix}} = \underset{\boldsymbol{b}}{\begin{pmatrix} 12 \\ 10 \end{pmatrix}} \quad \text{means} \quad \begin{aligned} 3x+4y&=12 \\ 2x-5y&=10 \end{aligned}$$

A system of linear equations can be represented as a single matrix equation.

Ax = ***b*** ***A*** is called the "***coefficient matrix***". In this example: $\begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}$

Like a regular equation, if we can "divide" by ***A*** we can find ***x***. But how do we "divide" by a matrix? That's coming up. Meanwhile, a guy named Cramer figured out that:

$$x = \frac{\begin{vmatrix} 12 & 4 \\ 10 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & -5 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 3 & 12 \\ 2 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & -5 \end{vmatrix}}$$

Cramer's Rule:

For any general 2x2 system $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Cramer's Rule can also be extended to larger systems. For a 3x3 system:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} j \\ k \\ l \end{pmatrix}$$

the solutions are: (can you guess?)

$$x=\frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \quad y=\frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \quad z=\frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

What happens if the determinant of the "coefficient matrix" is zero?

If the determinant of the coefficient matrix = 0, the system has **NO SOLUTION**!

Wow! Easy to find out whether there is a solution before you do all the work!
...and computers can do the calculations really easily for you!

Homework: 3.7 p. 207 #4-34 by 3 and #40-44 even

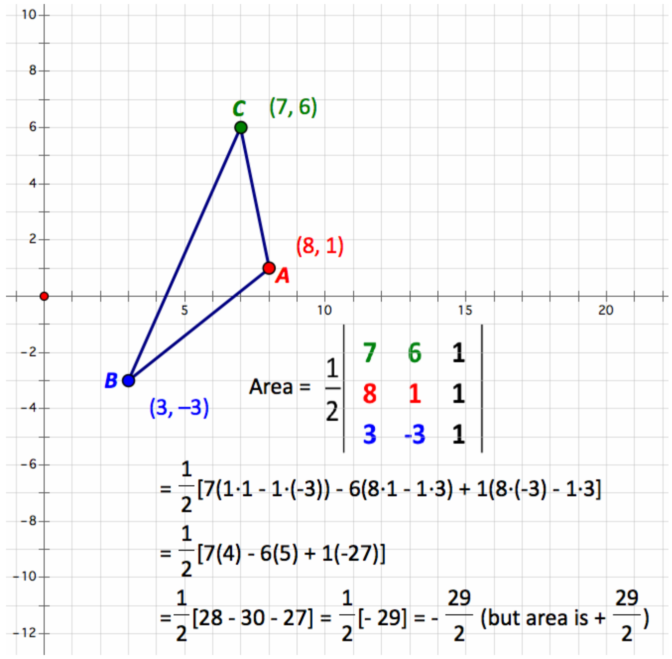
(Note for #44: The weight of a chemical compound is the sum of the weights of the atoms that make it up. A symbol like H_2O represents two hydrogen atoms and one oxygen atom.)

Another application of determinants

You can calculate the area of a triangle from it's vertices!

The area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by the absolute value of:

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$



You try: What is the area of a triangle with vertices (2, 3), (-5, 1) and (6, 8)?