Present 18F.1 #2gh & 18F.2 #1bc, 2

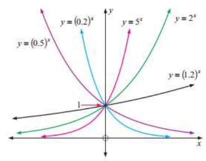
Chapter 19

Derivatives of exponential and logarithmic functions

- **A** Exponential e
- **B** Natural logarithms
- **C** Derivatives of logarithmic functions
- **D** Applications

A EXPONENTIAL e

Recall the graphs of exponential functions of the form $y = a^x$



For base > 1 the function ***increases,*** for 0 < base < 1 the function ***decreases***. Let's find the derivative of this function. Since we don't have a formula yet, we need to start from first principles:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{a^{(x+h)}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = \lim_{h\to 0}\frac{a^x(a^h-1)}{h}$$
$$= a^x \lim_{h\to 0}\frac{(a^h-1)}{h}$$

Now we notice something interesting:

$$\lim_{h\to 0}\frac{(a^h-1)}{h} = \lim_{h\to 0}\frac{(a^{0+h}-a^0)}{h} = f'(0)$$

So we can rewrite our result as:

Derivative of an exponential
If $f(x) = a^x$ then $f'(x) = f'(0)a^x$

As it turns out, the function e^x has a slope of one at $x = 0$. So it has the very special property that:

$$\frac{d}{dx}(e^x) = e^x$$

It is a function whose derivative is itself! That is, the ***value of the function*** at any point x is also the ***slope of the function*** at that point! Pretty natural, eh?

From the chain rule, we can also evaluate derivatives of functions of the form $e^{f(x)}$

Function	*Derivative*
e^x	e^x
$e^{f(x)}$	$e^{f(x)} \times f'(x)$

We can now put this together with previous ideas to answer some questions:

Find the gradient function for y equal to:

a $2e^x + e^{-3x}$ **b** x^2e^{-x} **c** $\dfrac{e^{2x}}{x}$

a If $y = 2e^x + e^{-3x}$ then $\dfrac{dy}{dx} = 2e^x + e^{-3x}(-3)$
$= 2e^x - 3e^{-3x}$

b If $y = x^2e^{-x}$ then $\dfrac{dy}{dx} = 2xe^{-x} + x^2e^{-x}(-1)$ {product rule}
$= 2xe^{-x} - x^2e^{-x}$

c If $y = \dfrac{e^{2x}}{x}$ then $\dfrac{dy}{dx} = \dfrac{e^{2x}(2)x - e^{2x}(1)}{x^2}$ {quotient rule}
$= \dfrac{e^{2x}(2x-1)}{x^2}$

Find the gradient function for y equal to: **a** $(e^x-1)^3$ **b** $\dfrac{1}{\sqrt{2e^{-x}+1}}$

a $y = (e^x-1)^3$
$= u^3$ where $u = e^x - 1$
$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}
$= 3u^2\dfrac{du}{dx}$
$= 3(e^x-1)^2 \times e^x$
$= 3e^x(e^x-1)^2$

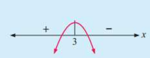
b $y = (2e^{-x}+1)^{-\frac{1}{2}}$
$= u^{-\frac{1}{2}}$ where $u = 2e^{-x}+1$
$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}
$= -\frac{1}{2}u^{-\frac{3}{2}}\dfrac{du}{dx}$
$= -\frac{1}{2}(2e^{-x}+1)^{-\frac{3}{2}} \times 2e^{-x}(-1)$
$= e^{-x}(2e^{-x}+1)^{-\frac{3}{2}}$

Find the exact position and nature of the stationary points of $y = (x-2)e^{-x}$.

$\dfrac{dy}{dx} = (1)e^{-x} + (x-2)e^{-x}(-1)$ {product rule}
$= e^{-x}(1-(x-2))$
$= \dfrac{3-x}{e^x}$ where e^x is positive for all x.

So, $\dfrac{dy}{dx} = 0$ when $x = 3$.

The sign diagram of $\dfrac{dy}{dx}$ is:



$\therefore$ at $x = 3$ we have a local maximum.

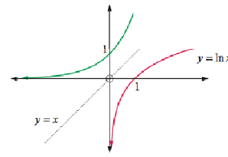
But when $x = 3$, $y = (1)e^{-3} = \dfrac{1}{e^3}$

$\therefore$ the local maximum is $(3, \dfrac{1}{e^3})$.

B NATURAL LOGARITHMS

A quick review:

- $e^x = a \Leftrightarrow x = \ln a$
- $e^{\ln a} = a$ and $\ln e^a = a$
- $\ln x$ is the inverse of e^x
- Domain of e^x is $x \in \mathbb{R}$, Range of e^x is
- Domain of $\ln x$ is $x > 0$, Range of $\ln x$ is $y \in \mathbb{R}$



Rules of Logarithms (any base)

For $a > 0, \quad b > 0$
- $\ln(ab) = \ln a + \ln b$
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln(a^n) = n \ln a$ **Note:** $\ln e^n = n$

Notice also that:
- $\ln 1 = 0$ and $\ln e = 1$
- $\ln\left(\frac{1}{a}\right) = -\ln a$
- $\log_b a = \frac{\ln a}{\ln b}, \quad b \neq 1$

Let's take another look at the derivative of a^x

Notice that a^x can be written as $(e^{\ln a})^x$ or $e^{x \ln a}$

Our understanding of the derivative of e^x tells us that

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a}) = \ln a \cdot e^{x \ln a} = \ln a \cdot a^x = a^x \cdot \ln a$$

From our previous formula, this implies that if $f(x) = a^x$, $f'(0) = \ln(a)$. In words, the slope of a^x at $x = 0$ is $\ln(a)$!

Let's work with logs a bit before moving on to their derivatives

Find algebraically, the exact points of intersection of $y = e^x - 3$ and $y = 1 - 3e^{-x}$. Check your solution using technology.

The functions meet where

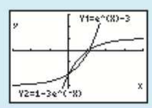
$$e^x - 3 = 1 - 3e^{-x}$$
$$\therefore \quad e^x - 4 + 3e^{-x} = 0$$
$$\therefore \quad e^{2x} - 4e^x + 3 = 0 \quad \{\text{multiplying each term by } e^x\}$$
$$\therefore \quad (e^x - 1)(e^x - 3) = 0$$
$$\therefore \quad e^x = 1 \text{ or } 3$$
$$\therefore \quad x = \ln 1 \text{ or } \ln 3$$
$$\therefore \quad x = 0 \text{ or } \ln 3$$

When $x = 0$, $y = e^0 - 3 = -2$

When $x = \ln 3$, $e^x = 3 \quad \therefore \quad y = 3 - 3 = 0$

$\therefore$ the functions meet at $(0, -2)$ and at $(\ln 3, 0)$.

GRAPHING PACKAGE



Consider the function $y = 2 - e^{-x}$.

a Find the x-intercept. **b** Find the y-intercept.

c Show algebraically that the function is increasing for all x.

d Show algebraically that the function is concave down for all x.

e Use technology to help graph $y = 2 - e^{-x}$.

f Explain why $y = 2$ is a horizontal asymptote.

a The x-intercept occurs when $y = 0$, $\therefore \; e^{-x} = 2$

$$\therefore \; -x = \ln 2$$
$$\therefore \; x = -\ln 2$$

$\therefore$ the x-intercept is $-\ln 2 \approx -0.69$

b The y-intercept occurs when $x = 0$

$\therefore \; y = 2 - e^0 = 2 - 1 = 1$

c $\frac{dy}{dx} = 0 - e^{-x}(-1) = e^{-x} = \frac{1}{e^x}$

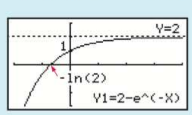
Now $e^x > 0$ for all x, so $\frac{dy}{dx} > 0$ for all x

$\therefore$ the function is increasing for all x.

d $\frac{d^2y}{dx^2} = e^{-x}(-1) = \frac{-1}{e^x}$ which is < 0 for all x

$\therefore$ the function is concave down for all x.

e



f As $x \to \infty$, $e^x \to \infty$

and $e^{-x} \to 0$

$\therefore \; y \to 2$ (below)

Hence, HA is $y = 2$.

19A: #1dhlop,2efg,3def,4,6,7bd (Derivatives of Exponents)
19B: #1cdgh,2,3befgh,4,5,7 (Working with Logs)

Quiz and Present 19A #1l, 2f,3f,6,7d 19B #3f,h,4c,7 as time allows

C DERIVATIVES OF LOGARITHMIC FUNCTIONS

What is the derivative of ln*x*?

$y = \ln x$ is equivalent to $e^y = x$

$$\frac{d}{dx}\left(e^y\right) = \frac{d}{dx}(x) = 1$$

We use the chain rule here: $e^y \frac{dy}{dx} = 1$ or $\frac{dy}{dx} = \frac{1}{e^y}$

But $e^y = x$ so $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Extending this to arguments that are functions of *x* and using the chain rule, we get the more general case:

Derivative of ***natural*** logarithms

$$\boxed{\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}}$$

Can you generalize to logs of any base?

$$\log_a f(x) = \frac{\ln f(x)}{\ln a} = \frac{1}{\ln a}\ln f(x)$$

$$\frac{d}{dx}\left(\frac{1}{\ln a}\ln f(x)\right) = \frac{1}{\ln a}\cdot\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{\ln a \cdot f(x)}$$

Find the gradient function of: **a** $y = \ln(kx)$ where k is a constant **b** $y = \ln(1-3x)$ **c** $y = x^3 \ln x$

a If $y = \ln(kx)$ then $\frac{dy}{dx} = \frac{k}{kx} = \frac{1}{x}$

b If $y = \ln(1-3x)$

then $\frac{dy}{dx} = \frac{-3}{1-3x} = \frac{3}{3x-1}$

c If $y = x^3 \ln x$

then $\frac{dy}{dx} = 3x^2 \ln x + x^3\left(\frac{1}{x}\right)$ {product rule}

$= 3x^2 \ln x + x^2$

$= x^2(3\ln x + 1)$

$\ln(kx) = \ln k + \ln x$
$= \ln x +$ constant
so $\ln(kx)$ and $\ln x$
both have derivative $\frac{1}{x}$.

Of course, this is usually going happen in a larger context.

Differentiate with respect to x:
a $y = \ln(xe^{-x})$ **b** $y = \ln\left[\frac{x^2}{(x+2)(x-3)}\right]$

a If $y = \ln(xe^{-x})$ then $y = \ln x + \ln e^{-x}$ {log of a product law}

$\therefore\ y = \ln x - x$ {$\ln e^a = a$}

Differentiating with respect to x, we get $\frac{dy}{dx} = \frac{1}{x} - 1$

b If $y = \ln\left[\frac{x^2}{(x+2)(x-3)}\right]$ then $y = \ln x^2 - \ln[(x+2)(x-3)]$

$= 2\ln x - [\ln(x+2) + \ln(x-3)]$

$= 2\ln x - \ln(x+2) - \ln(x-3)$

$\therefore\ \frac{dy}{dx} = \frac{2}{x} - \frac{1}{x+2} - \frac{1}{x-3}$

19C: #1jklmn,2ghi,3fghi,4,6 (Derivatives of Logs)
QB: 9*(a-d.iii)

D APPLICATIONS

19D: #1-21odd (Derivatives of Exponents and Logs)

Chapter 20

Derivatives of trigonometric functions

A Derivatives of trigonometric functions
B Optimisation with trigonometry

A DERIVATIVES OF CIRCULAR FUNCTIONS

Find the derivative of sin*x* from first principles.
A hint: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{[\text{illegible}] - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \frac{\sin x([\text{illegible}]}{h} + \lim_{h \to 0} \cos x \left(\frac{\sin h}{h}\right) \\
&= \lim_{h \to 0} -2 \sin x \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \frac{\sin\left(\frac{h}{2}\right)}{2} + \cos x \times \lim_{h \to 0} \frac{\sin h}{h} \\
&= -2 \sin x \times 1 \times 0 + \cos x \times 1 \\
&= \cos x
\end{aligned}
$$

There is an alternative derivation in the text using the identity:

$$\sin S - \sin D = 2 \cos\left(\tfrac{S+D}{2}\right) \sin\left(\tfrac{S-D}{2}\right)$$

Find the derivative of cos x (Hint: cos x = sin (x + π/2))

Find the derivative of tan x

Don't forget the chain rule (ever!)

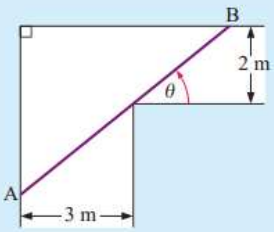
Summary: ***for x in radians:***

Function	Derivative
$\sin[f(x)]$	$\cos[f(x)]\, f'(x)$
$\cos[f(x)]$	$-\sin[f(x)]\, f'(x)$
$\tan[f(x)]$	$\sec^2[f(x)]\, f'(x)$

B OPTIMISATION WITH TRIGONOMETRY

These are just applications of derivatives that involve trig functions. Focus on understanding the underlying meaning of things. Pay attention to details, but absorb the meaning from the context.

Two corridors meet at right angles and are 2 m and 3 m wide respectively. θ is the angle marked on the given figure. [AB] is a thin metal tube which must be kept horizontal and cannot be bent as it moves around the corner from one corridor to the other.



a Show that the length AB is given by $L = \dfrac{3}{\cos\theta} + \dfrac{2}{\sin\theta}$.

b Show that $\dfrac{dL}{d\theta} = 0$ when $\theta = \tan^{-1}\left(\sqrt[3]{\tfrac{2}{3}}\right) \approx 41.1^o$.

c Find L when $\theta = \tan^{-1}\left(\sqrt[3]{\tfrac{2}{3}}\right)$ and comment on the significance of this value.

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a $\cos\theta = \dfrac{3}{a}$ and $\sin\theta = \dfrac{2}{b}$ so $a = \dfrac{3}{\cos\theta}$ and $b = \dfrac{2}{\sin\theta}$

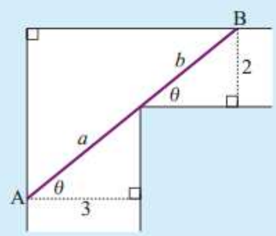
$\therefore \quad L = a + b = \dfrac{3}{\cos\theta} + \dfrac{2}{\sin\theta}$

b $L = 3[\cos\theta]^{-1} + 2[\sin\theta]^{-1}$

$$\therefore \quad \frac{dL}{d\theta} = -3[\cos\theta]^{-2} \times (-\sin\theta) - 2[\sin\theta]^{-2} \times \cos\theta$$

$$= \frac{3\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$

$$= \frac{3\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta}$$



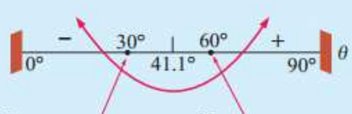
Thus $\dfrac{dL}{d\theta} = 0 \;\Leftrightarrow\; 3\sin^3\theta = 2\cos^3\theta$

$\therefore \quad \tan^3\theta = \frac{2}{3}$

$\therefore \quad \tan\theta = \sqrt[3]{\tfrac{2}{3}}$

and so $\theta = \tan^{-1}\left(\sqrt[3]{\tfrac{2}{3}}\right) \approx 41.1^o$

c *Sign diagram of* $\dfrac{dL}{d\theta}$:



$\dfrac{dL}{d\theta} \approx -4.93 < 0, \quad \dfrac{dL}{d\theta} \approx 9.06 > 0$

Thus, AB is minimised when $\theta \approx 41.1^o$. At this time $L \approx 7.02$ metres, so if we ignore the width of the rod then the greatest length of rod able to be horizontally carried around the corner is 7.02 m.

20A: #1ghi,2ijkl,3ijkl,5-11 odd,12 (Derivatives of trig functions)
20B: #1,3,5 (Derivatives of trig functions)
QB: 1*,35*a-c(trig&exp) ,37*a-d(trig),45a

Pace yourself through these problems. Do them all. Do not do them all at once. If you can't do one, sit with it for a while. Come back to it. Be persistent.