Problems from Chapter 20. HW was
20A: #1ghi,2ijkl,3ijkl,5-11 odd,12 (Derivatives of trig functions)
20B: #1,3,5 (Derivatives of trig functions)
QB: 1*,35*a-c(trig&exp) ,37*a-d(trig),45a

Present 20A #7,9,11
20B #1,3,5
QB #1, 35, 37

1. The point P $(\frac{1}{2}, 0)$ lies on the graph of the curve of $y = \sin(2x - 1)$.

Find the gradient of the tangent to the curve at P.

Working:	
	Answer: ....

(Total 4 marks)

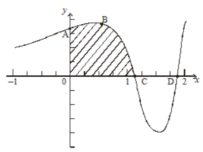
1. $y = \sin(2x - 1)$

$\frac{dy}{dx} = 2\cos(2x - 1)$ (A1)(A1)

At $\left(\frac{1}{2}, 0\right)$, the gradient of the tangent $= 2\cos 0$ (A1)

$= 2$ (A1) (C4)

35. The diagram below shows a sketch of the graph of the function $y = \sin(e^x)$ where $-1 \le x \le 2$, and x is in **radians**. The graph cuts the y-axis at A, and the x-axis at C and D. It has a maximum point at B.



(a) Find the coordinates of A. **(2)**

(b) The coordinates of C may be written as $(\ln k, 0)$. Find the **exact** value of k. **(2)**

(c) (i) Write down the y-coordinate of B.

(ii) Find $\frac{dy}{dx}$.

(iii) Hence, show that at B, $x = \ln\frac{\pi}{2}$.

(6)

35. (a) At A, $x = 0 \Rightarrow y = \sin(e^0) = \sin(1)$ (M1)
$\Rightarrow$ coordinates of A = (0,0.841) (A1)
OR
A(0, 0.841) (G2)2

(b) $\sin(e^x) = 0 \Rightarrow e^x = \pi$ (M1)
$\Rightarrow x = \ln\pi$ (or $k = \pi$) (A1)
OR
$x = \ln\pi$ (or $k = \pi$) (A2) 2

(c) (i) Maximum value of sin function = 1 (A1)

(ii) $\frac{dy}{dx} = e^x\cos(e^x)$ (A1)(A1)

Note: *Award (A1) for $\cos(e^x)$ and (A1) for e^x.*

(iii) $\frac{dy}{dx} = 0$ at a maximum (R1)

$e^x\cos(e^x) = 0$

$\Rightarrow e^x = 0$ (impossible) or $\cos(e^x) = 0$ (M1)

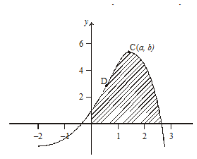
$\Rightarrow e^x = \frac{\pi}{2} \Rightarrow x = \ln\frac{\pi}{2}$ (A1)(AG) 6

37. Consider the function $f(x) = \cos x + \sin x$.

(a) (i) Show that $f(-\frac{\pi}{4}) = 0$.

(ii) Find in terms of π, the smallest **positive** value of x which satisfies $f(x) = 0$. **(3)**

The diagram shows the graph of $y = e^x(\cos x + \sin x)$, $-2 \le x \le 3$. The graph has a maximum turning point at C(a, b) and a point of inflexion at D.



(b) Find $\frac{dy}{dx}$. **(3)**

(c) Find the **exact** value of a and of b. **(4)**

(d) Show that at D, $y = \sqrt{2}e^{\frac{\pi}{4}}$. **(5)**

37. (a) (i) $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ (A1)

therefore $\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right) = 0$ (AG)

(ii) $\cos x + \sin x = 0 \Rightarrow 1 + \tan x = 0$
$\Rightarrow \tan x = -1$ (M1)
$x = \frac{3\pi}{4}$ (A1)

Note: *Award (A0) for 2.36.*

OR

$x = \frac{3\pi}{4}$ (G2) 3

(b) $y = e^x(\cos x + \sin x)$

$\frac{dy}{dx} = e^x(\cos x + \sin x) + e^x(-\sin x + \cos x)$ (M1)(A1)(A1) 3

$= 2e^x\cos x$

(c) $\frac{dy}{dx} = 0$ for a turning point $\Rightarrow 2e^x\cos x = 0$ (M1)

$\Rightarrow \cos x = 0$ (A1)

$\Rightarrow x = \frac{\pi}{2} \Rightarrow a = \frac{\pi}{2}$ (A1)

$y = e^{\frac{\pi}{2}}(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}) = e^{\frac{\pi}{2}}$

$b = e^{\frac{\pi}{2}}$ (A1) 4

Note: *Award (M1)(A1)(A0)(A0) for $a = 1.57$, $b = 4.81$.*

(d) At D, $\frac{d^2y}{dx^2} = 0$ (M1)

$2e^x\cos x - 2e^x\sin x = 0$ (A1)

$2e^x(\cos x - \sin x) = 0$

$\Rightarrow \cos x - \sin x = 0$ (A1)

$\Rightarrow x = \frac{\pi}{4}$ (A1)

$\Rightarrow y = e^{\frac{\pi}{4}}(\cos\frac{\pi}{4} + \sin\frac{\pi}{4})$ (A1)

$= \sqrt{2}e^{\frac{\pi}{4}}$ (AG) 5

Chapter 21
Integration

Chapter 22
Applications of integration

A ANTIDIFFERENTIATION

Elliot is cruising along a straight road at 90 ft/sec when his radar detector beeps. He slams on his brakes, decelerating at a rate of 20 ft/sec^2. a) How long does it take him to get to the 20 mph speed limit (30 ft/sec)? b) How far does he travel in that time?

a) $30 = -20t + 90$ $t = 3$ seconds

b) We need a function that describes position as a function of time. Recall that velocity is the derivative of position. So first we want to find a function whose derivative is $v(t) = -20t + 90$

$$s(t) = -10t^2 + 90t + c$$

> The ***antiderivative*** of a function $f(x)$ is that function whose derivative is $f(x)$.

The antiderivative of f is written as F with the property that $F'(x) = f(x)$

We find antiderivatives by
a) thinking in reverse from differentiation or
b) integration (we will look more at this later)

Some examples:

1. Solve the following *antiderivative* questions. In other words, find F, g, and S:

(a) $F'(x) = 4x^3$ **(b)** $g'(t) = 10\cos(5t)$ **(c)** $\frac{dS}{du} = \frac{1}{2}e^u + \frac{1}{2}e^{-u}$

Did you notice that there is more than one correct answer to each question?

B THE FUNDAMENTAL THEOREM OF CALCULUS

Back to our question. We found that the ***position*** of the object is given by:

$$s(t) = -10t^2 + 90t + c$$

Since Elliot has not backtracked at all, we can find the distance travelled by looking at his position at $t = 0$ and his position at $t = 3$:

$$s(0) = -10(0)^2 + 90(0) + c = c \qquad s(3) = -10(3)^2 + 90(3) + c = 180 + c$$

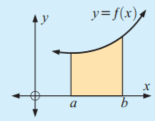
If we take the difference, we have the distance travelled:

$$s(3) - s(0) = 180 + c - c = 180 \text{ feet}$$

Let's look at this visually

Notice that the distance traveled is the area under the curve since we are summing up bits of rate times time. Areas under curves can represent lots of things!

If $f(x)$ is a continuous positive function on an interval $a \leqslant x \leqslant b$ then the area under the curve between $x = a$ and $x = b$ is $\displaystyle\int_a^b f(x)\,dx$.



One can calculate these areas by using infinite summations of rectangles or trapezoids but it is rather complex. We will state (without proving) a very important result:

The Fundamental Theorem of Calculus (Part I)

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

In other words instead of evaluating an infinite sum using limits, we can simply evaluate the difference of the antiderivative of the function evaluated at the two endpoints of the interval.

This is a ***huge*** simplification! From this theorem, other properties can be proven:

- $\displaystyle\int_a^a f(x)\,dx = 0$
- $\displaystyle\int_a^b c\,dx = c(b-a)$ $\quad\{c \text{ is a constant}\}$
- $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\displaystyle\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Understanding that integrals are just (signed) ***areas under a curve*** helps you remember these.

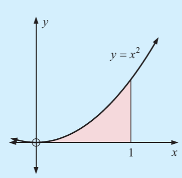
Let's try a couple:

Use the fundamental theorem of calculus to find the area:

a between the x-axis and $y = x^2$ from $x = 0$ to $x = 1$

b between the x-axis and $y = \sqrt{x}$ from $x = 1$ to $x = 9$.

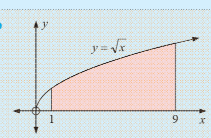
a

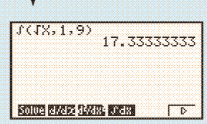


$f(x) = x^2$ has antiderivative $F(x) = \dfrac{x^3}{3}$

$$\begin{aligned}\therefore\ \text{the area } &= \int_0^1 x^2\,dx \\ &= F(1) - F(0) \\ &= \tfrac{1}{3} - 0 \\ &= \tfrac{1}{3} \text{ units}^2\end{aligned}$$

b





$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

$$\begin{aligned}\therefore\ \text{the area } &= \int_1^9 x^{\frac{1}{2}}\,dx \\ &= F(9) - F(1) \\ &= \tfrac{2}{3} \times 27 - \tfrac{2}{3} \times 1 \\ &= 17\tfrac{1}{3} \text{ units}^2\end{aligned}$$

21A: #1-3 (Finding antiderivatives)
21B: #1,2defg,3,4 (Fundamental theorem)

C INTEGRATION

Let's look again at finding the area under a curve over a specific interval.

But the Fundamental Theorem of Calculus is more powerful than that. We can use it to relate a function and its antiderivative, even if there is no specific interval under consideration. The Fundamental Theorem implies that:

$$\text{if } F'(x) = f(x) \text{ then } \int f(x)\,dx = F(x) + c.$$

In words, the ***indefinite integral*** of a function is given by the antiderivative of the function plus some constant of integration. The function $f(x)$ is called the ***integrand***.

Example: Consider the function $f(x) = x^2$

The antiderivative of this function is $F(x) = \frac{x^3}{3} + c$ since it's derivative gives $f(x)$

We write this as the ***integral*** of f or $\int x^2\,dx = \frac{x^3}{3} + c$

Notice that the same rules apply as for definite integrals, except that there is always a constant of integration. Another example.

Find $\int (2x^2 - 4x + 5)dx$

$$\begin{aligned}
&= \int 2x^2\,dx - \int 4x\,dx + \int 5dx \\
&= 2\int x^2\,dx - 4\int x\,dx + \int 5dx \\
&= \left(2\frac{x^3}{3} + c_1\right) - \left(4\frac{x^2}{2} + c_2\right) + (5x + c_3) \\
&= \frac{2x^3}{3} - 2x^2 + 5x + c
\end{aligned}$$

Check your work! If you differentiate your result, you should get the integrand!

In the HW you will practice seeing the relationship between simple integrals and their associated derivatives.

C INTEGRATION

Since we know a lot about differentiation, we can work in reverse to figure out a lot of integrals! Recall the following properties of differentiation:

Function	*Derivative*	*Name*
c, a constant	0	
$mx+c$, m and c are constants	m	
x^n	nx^{n-1}	**power rule**
$cu(x)$	$cu'(x)$	
$u(x)+v(x)$	$u'(x)+v'(x)$	**addition rule**
$u(x)v(x)$	$u'(x)v(x)+u(x)v'(x)$	**product rule**
$\dfrac{u(x)}{v(x)}$	$\dfrac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$	**quotient rule**
$y=f(u)$ where $u=u(x)$	$\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx}$	**chain rule**
e^x	e^x	
$e^{f(x)}$	$e^{f(x)}f'(x)$	
$\ln x$	$\dfrac{1}{x}$	
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$	
$[f(x)]^n$	$n[f(x)]^{n-1}f'(x)$	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\dfrac{1}{\cos^2 x}$	

These lead to the following integrals:

Function	*Integral*
k, a constant	$kx+c$
x^n	$\dfrac{x^{n+1}}{n+1}+c,\ n\neq -1$
e^x	e^x+c
$\dfrac{1}{x}$	$\ln x+c,\ x>0$
$\cos x$	$\sin x+c$
$\sin x$	$-\cos x+c$

$\int k\,dx = kx+c$

$\int x^n dx = \dfrac{x^{n+1}}{n+1}+c$ for $n\neq 1$

$\int e^x dx = e^x+c$

$\int \dfrac{1}{x}dx = \ln x + c$

$\int \cos x\,dx = \sin x + c$

$\int \sin x\,dx = -\cos x + c$

But beware of $x < 0$:

$\therefore \int \frac{1}{x}dx = \ln|x|+c$

Notice that because of the chain rule, these rules ***do not work*** directly when the argument is a ***function of x***. Don't forget about the constant of integration.

Using this in conjunction with what you know about differentiating, you can find many integrals. Let's try some:

Find: **a** $\displaystyle\int (x^3-2x^2+5)\,dx$ **b** $\displaystyle\int\left(\frac{1}{x^3}-\sqrt{x}\right)dx$

a $\displaystyle\int (x^3-2x^2+5)\,dx$

$= \dfrac{x^4}{4}-\dfrac{2x^3}{3}+5x+c$

b $\displaystyle\int\left(\frac{1}{x^3}-\sqrt{x}\right)dx$

$= \displaystyle\int (x^{-3}-x^{\frac{1}{2}})\,dx$

$= \dfrac{x^{-2}}{-2}-\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}}+c$

$= -\dfrac{1}{2x^2}-\frac{2}{3}x^{\frac{3}{2}}+c$

Always differentiate your result to see that you get the original integrand.

Integrate with respect to x:

a $2\sin x-\cos x$ **b** $-\dfrac{2}{x}+3e^x$

a $\displaystyle\int [2\sin x-\cos x]dx$

$= 2(-\cos x)-\sin x+c$

$= -2\cos x-\sin x+c$

b $\displaystyle\int\left[-\frac{2}{x}+3e^x\right]dx$

$= -2\ln x+3e^x+c$ provided $x>0$

Find: **a** $\displaystyle\int\left(3x+\frac{2}{x}\right)^2 dx$ **b** $\displaystyle\int\left(\frac{x^2-2}{\sqrt{x}}\right)dx$

a $\displaystyle\int\left(3x+\frac{2}{x}\right)^2 dx$

$= \displaystyle\int\left(9x^2+12+\frac{4}{x^2}\right)dx$

$= \displaystyle\int (9x^2+12+4x^{-2})\,dx$

$= \dfrac{9x^3}{3}+12x+\dfrac{4x^{-1}}{-1}+c$

$= 3x^3+12x-\dfrac{4}{x}+c$

b $\displaystyle\int\left(\frac{x^2-2}{\sqrt{x}}\right)dx$

$= \displaystyle\int\left(\frac{x^2}{\sqrt{x}}-\frac{2}{\sqrt{x}}\right)dx$

$= \displaystyle\int (x^{\frac{3}{2}}-2x^{-\frac{1}{2}})\,dx$

$= \dfrac{x^{\frac{5}{2}}}{\frac{5}{2}}-\dfrac{2x^{\frac{1}{2}}}{\frac{1}{2}}+c$

$= \frac{2}{5}x^2\sqrt{x}-4\sqrt{x}+c$

Expand parentheses and simplify to get a form you can integrate (when possible).

When you know any specific point on the curve that results from the integration, you can find the constant of integration.

Find $f(x)$ given that $f'(x)=x^3-2x^2+3$ and $f(0)=2$.

Since $f'(x)=x^3-2x^2+3$,

$f(x)=\displaystyle\int (x^3-2x^2+3)\,dx$

$\therefore\ f(x)=\dfrac{x^4}{4}-\dfrac{2x^3}{3}+3x+c$

But $f(0)=2$, so $0-0+0+c=2$ and so $c=2$

Thus $f(x)=\dfrac{x^4}{4}-\dfrac{2x^3}{3}+3x+2$

We may be given a second derivative, ***f''***. To find the original function, integrate twice. You will need a point on ***f*** to be specific on the second integration.

Find $f(x)$ given that $f''(x)=12x^2-4$, $f'(0)=-1$ and $f(1)=4$.

If $f''(x)=12x^2-4$

then $f'(x)=\dfrac{12x^3}{3}-4x+c$ {integrating with respect to x}

$\therefore\ f'(x)=4x^3-4x+c$

But $f'(0)=-1$ so $0-0+c=-1$ and so $c=-1$

Thus $f'(x)=4x^3-4x-1$

$\therefore\ f(x)=\dfrac{4x^4}{4}-\dfrac{4x^2}{2}-x+d$ {integrating again}

$\therefore\ f(x)=x^4-2x^2-x+d$

But $f(1)=4$ so $1-2-1+d=4$ and so $d=6$

Thus $f(x)=x^4-2x^2-x+6$

Remember, when we see $g(x)=\int f(x)dx$ the function ***f*** is the ***derivative*** of ***g***

21C.1: #2-11 (Integration with & without boundary conditions)
21C.2: #1-8 last col, 9 & 10 (Integration practice)
QB: 36,42,48

D INTEGRATING $f(ax+b)$

This is a basic extension of integrating *f*(*x*). Notice that the derivative of *ax* + *b* is just *a*. So when we integrate a function of *ax* + *b*, we will need to divide by *a*. This is a consequence of the chain rule.

Apart from noticing this situation, there is nothing new here. It applies to any function.

Function	*Integral*
e^{ax+b}	$\frac{1}{a}e^{ax+b}+c$
$(ax+b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c, \quad n \neq -1$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b)+c, \quad ax+b>0$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)+c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)+c$

Find:
a $\int(2x+3)^4\,dx$
b $\int\frac{1}{\sqrt{1-2x}}\,dx$

a $\int(2x+3)^4\,dx$

$= \frac{1}{2}\times\frac{(2x+3)^5}{5}+c$

$= \frac{1}{10}(2x+3)^5+c$

b $\int\frac{1}{\sqrt{1-2x}}\,dx$

$= \int(1-2x)^{-\frac{1}{2}}\,dx$

$= \frac{1}{-2}\times\frac{(1-2x)^{\frac{1}{2}}}{\frac{1}{2}}+c$

$= -\sqrt{1-2x}+c$

Find: **a** $\int(2e^{2x}-e^{-3x})\,dx$ **b** $\int\frac{4}{1-2x}\,dx$

a $\int(2e^{2x}-e^{-3x})\,dx$

$= 2(\frac{1}{2})e^{2x}-(\frac{1}{-3})e^{-3x}+c$

$= e^{2x}+\frac{1}{3}e^{-3x}+c$

b $\int\frac{4}{1-2x}\,dx = 4\int\frac{1}{1-2x}\,dx$

$= 4\left(\frac{1}{-2}\right)\ln|1-2x|+c$

$= -2\ln|1-2x|+c$

Notice that integrals involving squares of trig functions cannot be done with the power rule unless there are other factors to offset the effects of the chain rule. However, we can often move forward by rewriting using the double angle formula. Recall that:

$$\sin^2\theta = \tfrac{1}{2}-\tfrac{1}{2}\cos(2\theta) \quad \text{or} \quad \cos^2\theta = \tfrac{1}{2}+\tfrac{1}{2}\cos(2\theta).$$

(These are shown in a different form on your formula sheet)

These allow us to tackle integrals like:

Integrate $(2-\sin x)^2$.

$\int(2-\sin x)^2\,dx$

$= \int(4-4\sin x+\sin^2 x)\,dx$

$= \int\left(4-4\sin x+\frac{1}{2}-\frac{1}{2}\cos(2x)\right)dx$

$= \int\left(\frac{9}{2}-4\sin x-\frac{1}{2}\cos(2x)\right)dx$

$= \frac{9}{2}x+4\cos x-\frac{1}{2}\times\frac{1}{2}\sin(2x)+c$

$= \frac{9}{2}x+4\cos x-\frac{1}{4}\sin(2x)+c$

21D: #1begh,2beh,3,4,5bcef,6cfi,7bc,8-13 (Integrating f(ax + b))
QB: 21,23,24b,28,33a,38,46b,47(power),30a,39*a

E DEFINITE INTEGRALS

Recall that we can use the fundamental theorem of calculus to evaluate ***definite intervals***. Note that in order to evaluate $\int_a^b f(x)dx$ ***the function f must be continuous on the interval*** *$a \le x \le b$*

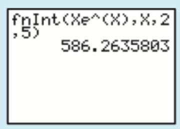
a and *b* are, appropriately, called the ***limits of integration***. Because the constant of integration will cancel out, we do not use them with definite integrals.

IB uses the notation $\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$

Find $\int_1^3 (x^2+2)\,dx$.	$\int_1^3 (x^2+2)\,dx$ $= \left[\frac{x^3}{3} + 2x\right]_1^3$ $= \left(\frac{3^3}{3} + 2(3)\right) - \left(\frac{1^3}{3} + 2(1)\right)$ $= (9+6) - (\frac{1}{3} + 2)$ $= 12\frac{2}{3}$

Some integrals cannot be evaluated using analytical techniques. Consider, for example,

Evaluate $\int_2^5 xe^x\,dx$ to an accuracy of 4 significant figures.



$\int_2^5 xe^x\,dx \approx 586.3$

Finding a function that differentiates to give xe^x (it's antiderivative) is no trivial matter. But if we have access to a good calculator and if the integral is definite, we can approximate the integral with the calculator:
MATH/fnInt (option 9)
The parameters are:
fnInt(<function>, <variable>, <lower limit>, <upper limit>)

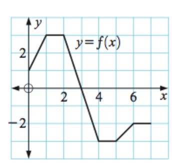
Recall that definite integrals have various properties that we have already proven. Use these liberally (in both directions) to simplify your life.

- $\int_a^b [-f(x)]\,dx = -\int_a^b f(x)\,dx$
- $\int_a^b cf(x)dx = c\int_a^b f(x)dx, \quad c$ is any constant
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)dx$
- $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Evaluate the following integrals using area interpretation:

a $\int_0^3 f(x)\,dx$ **b** $\int_3^7 f(x)\,dx$

c $\int_2^4 f(x)\,dx$ **d** $\int_0^7 f(x)\,dx$



Given that $\int_{-1}^1 f(x)\,dx = -4$, determine the value of:

a $\int_1^{-1} f(x)\,dx$ **b** $\int_{-1}^1 (2+f(x))\,dx$ **c** $\int_{-1}^1 2f(x)\,dx$

d k such that $\int_{-1}^1 kf(x)\,dx = 7$

21E.1: #1col2,2 (Integrals w/calculator)
21E.2: #2-10 even & 9 (Integrals w/calculator)
QB: 45b(trig),30b,32,41(exp),31(ln&trig)49(ln)

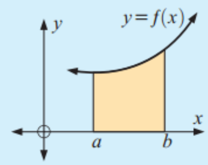
Thus ends your initial introduction to techniques of integration. Chapter 22 goes into some applications.

Chapter 22 Applications of integration

A FINDING AREAS BETWEEN CURVES

Integration is commonly used to find the area beneath a curve but above the *x*-axis. Indeed, this is a fundamental concept underlying the idea of integration.

If $f(x)$ is a continuous positive function on an interval $a \leqslant x \leqslant b$ then the area under the curve between $x = a$ and $x = b$ is $\displaystyle\int_a^b f(x)\,dx$.

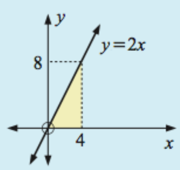


For many functions, we can use the Fundamental Theorem to calculate these types of areas:

Find the area of the region enclosed by $y = 2x$, the x-axis, $x = 0$ and $x = 4$ by using:

a a geometric argument **b** integration.

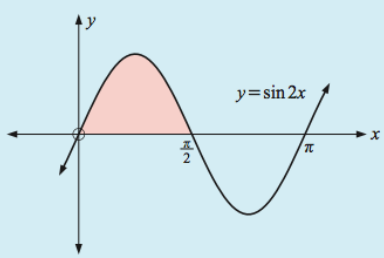
a



Area $= \frac{1}{2} \times 4 \times 8$
$= 16$ units^2

b Area $= \displaystyle\int_0^4 2x\,dx$
$= \left[x^2\right]_0^4$
$= 4^2 - 0^2$
$= 16$ units^2

Find the area enclosed by one arch of the curve $y = \sin 2x$.

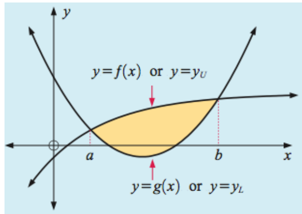


The period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$.

$\therefore$ the first positive x-intercept is $\frac{\pi}{2}$.

The required area $= \displaystyle\int_0^{\frac{\pi}{2}} \sin 2x\,dx$
$= \left[\frac{1}{2}(-\cos 2x)\right]_0^{\frac{\pi}{2}}$
$= -\frac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{2}}$
$= -\frac{1}{2}(\cos\pi - \cos 0)$
$= 1$ unit^2

This idea can be extended to find the area between two curves. Consider:



The "height" of the area at any value x is the difference between the two functions: $f(x) - g(x)$.

We need to be careful that this difference has the same sign over the interval being integrated. But in this case the sign of the individual functions is not what's important - just that the ***difference*** has the same sign.

Area Between Two Curves

The area between two curves f and g in the interval $[a, b]$ when $f(x) \geq g(x)$ in the interval is given by:

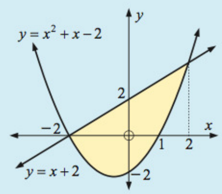
$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

You need to ***do the subtraction in the correct direction*** and ensure that f is above or on g over the entire interval. ***Sketch a graph***!

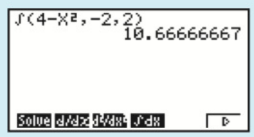
Find the area of the region enclosed by $y = x + 2$ and $y = x^2 + x - 2$.

$y = x + 2$ meets $y = x^2 + x - 2$ where

$$x^2 + x - 2 = x + 2$$
$$\therefore\ x^2 - 4 = 0$$
$$\therefore\ (x+2)(x-2) = 0$$
$$\therefore\ x = \pm 2$$



$$\text{Area} = \int_{-2}^{2} [y_U - y_L]\,dx$$
$$= \int_{-2}^{2} [(x+2) - (x^2 + x - 2)]\,dx$$
$$= \int_{-2}^{2} (4 - x^2)\,dx$$
$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$
$$= (8 - \tfrac{8}{3}) - (-8 + \tfrac{8}{3})$$
$$= 10\tfrac{2}{3} \text{ units}^2$$

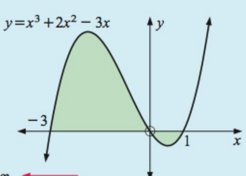


What would you do if f is ***not*** above g over the entire interval? You guessed it - break into two intervals.

Find the total area of the regions contained by $y = f(x)$ and the x-axis for $f(x) = x^3 + 2x^2 - 3x$.

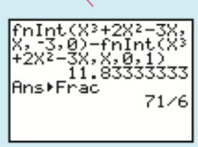
$$f(x) = x^3 + 2x^2 - 3x$$
$$= x(x^2 + 2x - 3)$$
$$= x(x-1)(x+3)$$

$\therefore$ $y = f(x)$ cuts the x-axis at 0, 1, and -3.



Total area

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\,dx - \int_{0}^{1} (x^3 + 2x^2 - 3x)\,dx$$
$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{0}^{1}$$
$$= (0 - -11\tfrac{1}{4}) - (-\tfrac{7}{12} - 0)$$
$$= 11\tfrac{5}{6} \text{ units}^2$$



22A.1: #1,2acehj,3,4 (Areas below curves (1 a-l))
22A.2: #2-16even (Areas between curves)
QB: 2d,9*(d.iv),16*d,17*(ab),25*(xdy),29*,35*de,37*e,39*b,43*,50*(trig),40*d,44d(rational)
Note: Many problems - use your discretion

Problems encountered in 22A?

B MOTION PROBLEMS

We've already seen that velocity is the derivative of position and acceleration is the derivative of velocity.

(Try Dramamine)

$$Velocity = \frac{\Delta Position}{\Delta Time} = \frac{d(Position)}{dt}$$
$$Acceleration = \frac{\Delta Velocity}{\Delta Time} = \frac{d(Velocity)}{dt} = \frac{d^2(Position)}{dt^2}$$

These ideas, in reverse, help us to find distance travelled, change in position, and change in velocity for some function that describes motion. A big idea in physics.

$$Distance\ traveled = |Area\ under\ V(t)| = \int |Velocity|\,dt = \int\int |Acceleration|\,dt$$
$$Change\ in\ Velocity = |Area\ under\ a(t)| = \int |Acceleration|\,dt$$

Be careful with the difference between ***distance travelled*** $= \int_{t_1}^{t_2} |v(t)|\,dt$

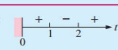
and ***total displacement (change in position)*** $= \int_{t_1}^{t_2} v(t)\,dt$

When the integrand changes sign and you want distance travelled, you need to:
a) Use the absolute value on your calculator or
b) break the integral into parts

The book describes a technique that involves finding different constants of integration for different intervals. I prefer to just break up the integral into different sections. In any case, you need to analyze the sign diagram of the integrand.

A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2\ \text{m s}^{-1}$.
a How far does P travel in the first 4 seconds of motion?
b Find the displacement of P after 4 seconds.

a $v(t) = s'(t) = t^2 - 3t + 2$
$= (t-1)(t-2)$ $\quad\therefore$ sign diagram of v is:



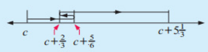
Since the signs change, P reverses direction at $t = 1$ and $t = 2$ secs.

Now $s(t) = \int (t^2 - 3t + 2)\,dt = \frac{t^3}{3} - \frac{3t^2}{2} + 2t + c$

Now $s(0) = c$ $\qquad s(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = c + \frac{5}{6}$

$s(2) = \frac{8}{3} - 6 + 4 + c = c + \frac{2}{3}$ $\qquad s(4) = \frac{64}{3} - 24 + 8 + c = c + 5\frac{1}{3}$

Motion diagram:



$\therefore$ total distance $= (c + \frac{5}{6} - c) + (c + \frac{5}{6} - [c + \frac{2}{3}]) + (c + 5\frac{1}{3} - [c + \frac{2}{3}])$
$= \frac{5}{6} + \frac{5}{6} - \frac{2}{3} + 5\frac{1}{3} - \frac{2}{3}$
$= 5\frac{2}{3}$ m

```
fnInt(abs(X²-3X+
2),X,0,4)
         5.666666667
```

b Displacement $=$ final position $-$ original position
$= s(4) - s(0)$
$= c + 5\frac{1}{3} - c$
$= 5\frac{1}{3}$ m $\qquad$ i.e., $5\frac{1}{3}$ m to the right.

```
fnInt(X²-3X+2,X,
0,4)
         5.333333333
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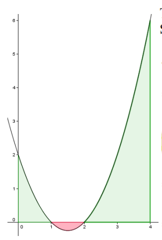


The velocity can be written as $v(t) = (t-1)(t-2)$ which changes sign at 1 & 2
So for the total distance traveled, we want

$$\int_0^1 (t^2 - 3t + 2)\,dt - \int_1^2 (t^2 - 3t + 2)\,dt + \int_2^4 (t^2 - 3t + 2)\,dt$$
$$= \frac{t^3}{3} - 3\frac{t^2}{2} + 2t\Big|_0^1 - \left(\frac{t^3}{3} - 3\frac{t^2}{2} + 2t\Big|_1^2\right) + \frac{t^3}{3} - 3\frac{t^2}{2} + 2t\Big|_2^4$$
$$\left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left\{\left(\frac{8}{3} - \frac{3\cdot 4}{2} + 4\right) - \left(\frac{1}{3} - \frac{3}{2} + 2\right)\right\} + \left(\frac{64}{3} - \frac{3\cdot 16}{2} + 8\right) - \left(\frac{8}{3} - \frac{3\cdot 4}{2} + 4\right)$$
$$= \frac{5}{6} - \left(\frac{2}{3} - \frac{5}{6}\right) + \frac{16}{3} - \frac{2}{3}$$
$$= \frac{10}{6} - \frac{4}{3} + \frac{16}{3} = \frac{5}{3} - \frac{4}{3} + \frac{16}{3} = \frac{17}{3}$$

Note calculator approach if it's available.

C PROBLEM SOLVING BY INTEGRATION

Nothing new here - just applying ideas of integration. Note that the ***amount*** that something changes is the integral of the ***rate at which it is changing***.

The primary issue here is getting the proper integral set up. Often you will have technology to help evaluate it. So focus on setting up the integrals.

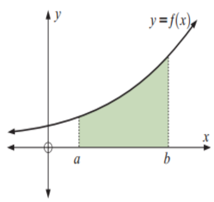
Draw a diagram!

22B.1: #1-3 (Distances from velocity)
22B.2: #2,4,6,7 (Motion problems)
22C: #1-4 (Problem solving)
QB: 3*,7*,11*(exp), 5,12*,15,20*(power)

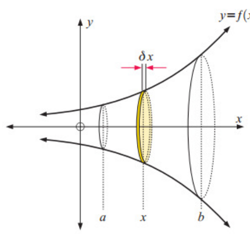
Before we begin, a little reminder about how to find the volume of a ***prism***.
What's a prism?

D SOLIDS OF REVOLUTION

A major use of integration is to calculate and ultimately optimize use of materials in manufacturing. Consider a Hershey's Kiss. The volume of chocolate can be calculated if we know the function, $f(x)$, that describes the contour of the side:



Imagine rotating the curve around the x-axis to create a ***solid of revolution***.



We can use an integral to sum up successive infinitesimally thin discs between a and b to get the volume created (Demo in HL pdf)

At any point x along the curve, the ***radius*** of the kiss is defined by the value of the function at that value of x. In other words, $r = f(x)$.

The volume of a single disc is approximately the area of its base times its height.

$A_{base} \cong \pi r^2 = \pi f(x)^2$ $\quad$ $h = \delta x$

So the volume of one disc can be approximated by:

$V = \pi r^2 \delta x$ and the total volume as the sum. If we take the limit as the thickness (δx) approaches zero, we get:

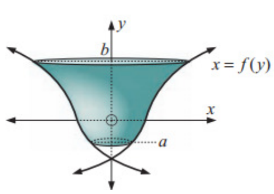
Volume of a solid defined by $f(x)$ rotated around the **x-axis** from a to b

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{b} \pi [f(x)]^2 \delta x = \pi \int_a^b f^2(x)dx = \pi \int_a^b y^2 dx$$

Use this idea to find a formula for the volume of a cone of height h and base radius r

The key, of course, is the complexity of the resulting integral which will depend on the defining function.

With a little imagination, one can also calculate the volume of a solid created by rotating a function around the y-axis. If the original function is given as $y = f(x)$, you will need to find the inverse function, $x = f(y)$ as the discs will be oriented horizontally:



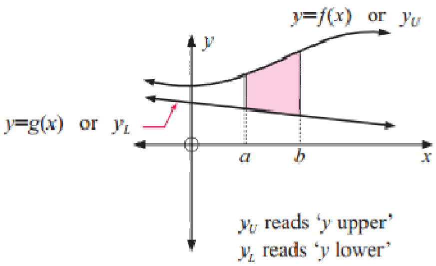
Volume of a solid defined by $f(x)$ rotated around the **y-axis** from a to b

$$V = \lim_{\delta y \to 0} \sum_{y=a}^{b} \pi [f(y)]^2 \delta y = \pi \int_a^b f^2(y)dy = \pi \int_a^b x^2 dx$$

Understand where these ideas come from and the formulas will be unnecssary.

VOLUMES FOR TWO DEFINING FUNCTIONS

The plot thickens as the volumes we want to calculate become more complex. To make something hollow, we can use two defining functions.

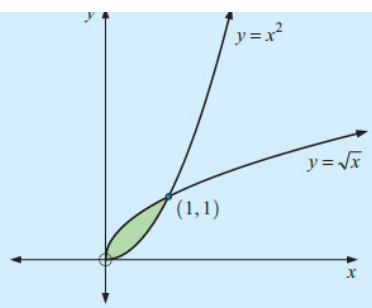


Similarly to what we saw calculating areas between two curves, rotating the upper function will generate one volume, while rotating the lower function will generate another volume to subtract from the first. This can be done in two steps or, sometimes more simply, by subtracting the functions ***before*** integrating.

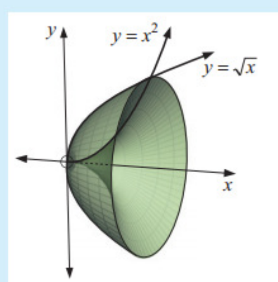
$$V=\pi\int_a^b f^2(x)dx-\pi\int_a^b g^2(x)dx=\pi\int_a^b[f^2(x)-g^2(x)]dx=\pi\int_a^b[y_U^2-y_L^2]dx$$

As was the case with areas, we need to pay close attention to which curve is the upper one. If that changes over the interval of integration, you must break the integral into different parts and change the direction of the subtraction.

Find the volume of revolution generated by revolving the region between $y=x^2$ and $y=\sqrt{x}$ about the x-axis.



$$\begin{aligned}
\text{Volume} &= \pi\int_0^1 \left(y_U^{\,2}-y_L^{\,2}\right)dx\\
&=\pi\int_0^1\left(\left(\sqrt{x}\right)^2-(x^2)^2\right)dx\\
&=\pi\int_0^1 (x-x^4)\,dx\\
&=\pi\left[\frac{x^2}{2}-\frac{x^5}{5}\right]_0^1\\
&=\pi\left(\left(\tfrac{1}{2}-\tfrac{1}{5}\right)-(0)\right)\\
&=\tfrac{3\pi}{10}\text{ units}^3
\end{aligned}$$



The primary issue here, like with problem solving, is getting the proper integral set up. Often you will have technology to help evaluate it. So focus on setting up the integrals.

22D.1: #1bdfh,2,4-8all (Volumes of rotation - one function)
22D.2: #2,3,4,6 (Volumes of rotation - two functions)
QB: 17*c, 26*,27*,36,42,48