Present 17G #4,7,10 17H #3b,4
QB 2(a-c),6,13,16(a-c),18

Chapter 18
Applications of differential calculus

A TIME RATE OF CHANGE

A derivative is a slope. A slope is a rate of change. ∴ a derivative is a rate of change.Often the context involves time. Your HW will explore derivatives in this context.

Hints: Write out the units in your problems! They are very helpful for keeping you on track.

Another point: ***velocity*** has direction, often positive or negative which needs to be interpreted in the context of the problem. The ***magnitude*** of velocity is ***speed***. That is,

Speed = |Velocity|

B GENERAL RATES OF CHANGE

A derivative is a slope. A slope is a rate of change. ∴ a derivative is a rate of change. Often the context **does not** involve time. Your HW will explore derivatives in this context.

Hints: Write out the units in your problems! They are very helpful for keeping you on track.

$\dfrac{dy}{dx}$ gives the **rate of change in y with respect to x.**

According to a psychologist, the ability of a person to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where t is the age in years, $5 \leqslant t \leqslant 18$.

a Find the rate of improvement in ability to understand spatial concepts when a person is: **i** 9 years old **ii** 16 years old.

b Explain why $\dfrac{dA}{dt} > 0$, $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

c Explain why $\dfrac{d^2A}{dt^2} < 0$, $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

a $A = \frac{1}{3}\sqrt{t} = \frac{1}{3}t^{\frac{1}{2}}$ $\quad \therefore \dfrac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \dfrac{1}{6\sqrt{t}}$

i When $t = 9$, $\dfrac{dA}{dt} = \frac{1}{18}$ $\quad\therefore$ the rate of improvement is $\frac{1}{18}$ units per year for a 9 year old.

ii When $t = 16$, $\dfrac{dA}{dt} = \frac{1}{24}$ $\quad\therefore$ the rate of improvement is $\frac{1}{24}$ units per year for a 16 year old.

b As $\sqrt{t}$ is never negative, $\dfrac{1}{6\sqrt{t}}$ is never negative

$\therefore \dfrac{dA}{dt} > 0$ for all $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age.

c $\dfrac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}}$ so $\dfrac{d^2A}{dt^2} = -\frac{1}{12}t^{-\frac{3}{2}} = -\dfrac{1}{12t\sqrt{t}}$

$\therefore \dfrac{d^2A}{dt^2} < 0$ for all $5 \leqslant t \leqslant 18$.

This means that while the ability to understand spatial concepts increases with time, the rate of increase slows down with age.

The cost of producing x items in a factory each day is given by

$$C(x) = \underbrace{0.000\,13x^3 + 0.002x^2}_{\text{cost of labour}} + \underset{\substack{\uparrow\\ \text{raw material costs}}}{5x} + \underset{\substack{\uparrow\\ \text{fixed or overhead costs such as heating, cooling, maintenance, rent}}}{2200}$$

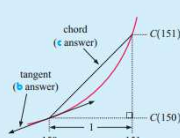
a Find $C'(x)$, which is called the marginal cost function.

b Find the marginal cost when 150 items are produced. Interpret this result.

c Find $C(151) - C(150)$. Compare this with the answer in **b**.

a The marginal cost function is
$C'(x) = 0.000\,39x^2 + 0.004x + 5$

b $C'(150) = \$14.38$
This is the rate at which the costs are increasing with respect to the production level x when 150 items are made per day. It gives an estimate of the cost for making the 151st item.



c $C(151) - C(150) \approx \$3448.19 - \$3433.75$
$\approx \$14.44$

This is the actual cost of making the 151st item each week, so the answer in **b** gives a good estimate.

18A: #1-4 (Time and motion)
18B: #1-7 odd (Other rates of change)
QB: 4*,8,10*,14,19

Again - this is core calculus material. Applications of derivatives such as these forms the basis for much future work.

Present 18A #3, 18B #3,5,7 QB #4,8,19 - save QB 10 & 14 for later

C MOTION IN A STRAIGHT LINE

Another common context is motion along a straight line. Consider the location, velocity, and acceleration of an object on the end of a spring for example.

Position

- A given problem involves an ***origin.*** It is generally the position of the object at $t = 0$, (not always). Know where it is in a given problem! In the spring problem, it can be the position of the spring at rest or it can be the location where the spring is attached to some support.
- Coordinate systems are intuitive or given. Up and right are > 0 left and down < 0.
- ***Position*** or ***Displacement*** (often we use the letter s) is ***the signed distance from the origin.*** It is a ***vector*** as it has magnitude and direction relative to the origin. Note the distinction between these and distance travelled.

Velocity

- ***Average velocity*** is the net change in position divided by elapsed time.

$$\frac{\Delta\text{position}}{\Delta\text{time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Note that ***velocity*** is also a ***vector*** as it has direction and magnitude relative to the origin. ***Speed***, on the other hand, is a ***scalar*** - the magnitude of velocity.

- ***Instantaneous velocity*** is the instantaneous rate of change of displacement vs time, better known as:

$$v(t) = \frac{ds}{dt} = s'(t)$$

Acceleration

- ***Average acceleration*** is the net change in velocity divided by elapsed time.

$$\frac{\Delta\text{velocity}}{\Delta\text{time}} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

Note that acceleration is also a ***vector*** as it has direction and magnitude relative to the origin.

- ***Instantaneous acceleration*** is the instantaneous rate of change of velocity vs time, better known as:

$$a(t) = \frac{dv}{dt} = v'(t) \text{ or } a(t) = \frac{d^2v}{dt^2} = v''(t)$$

- ***Acceleration***: be careful about interpretation.
  - > When v is **positive,** $a > 0$ is speeding up, $a < 0$ is slowing down
  - > When v is **negative,** $a > 0$ is slowing down, $a < 0$ is speeding up
  - > When $a = 0$ the velocity is constant.

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time t seconds. Find:

a the average velocity in the time interval from $t = 2$ to $t = 5$ seconds

b the average velocity in the time interval from $t = 2$ to $t = 2 + h$ seconds

c $\lim_{h\to 0} \frac{s(2+h) - s(2)}{h}$ and comment on its significance.

a average velocity

$$= \frac{s(5) - s(2)}{5 - 2}$$
$$= \frac{(15 - 25) - (6 - 4)}{3}$$
$$= \frac{-10 - 2}{3}$$
$$= -4 \text{ m s}^{-1}$$

b average velocity

$$= \frac{s(2+h) - s(2)}{2 + h - 2}$$
$$= \frac{3(2+h) - (2+h)^2 - 2}{h}$$
$$= \frac{6 + 3h - 4 - 4h - h^2 - 2}{h}$$
$$= \frac{-h - h^2}{h}$$
$$= -1 - h \text{ m s}^{-1} \text{ provided } h \neq 0$$

c $$\lim_{h\to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h\to 0} (-1 - h)$$
$$= -1 \text{ m s}^{-1}$$

This is the instantaneous velocity of the particle at time $t = 2$ seconds.

18C.1: #1&3 (Motion on a line)
18C.2: #1,3,5 (More motion on a line)

Present 18C.1 #3, 18C.2 #3,5

D SOME CURVE PROPERTIES

In math we often want to understand the properties of curves. What are some properties of curves that might be of interest?

- *y* - intercept(s)
- Zeros (*x* intercepts)
- Asymptotic behavior
- End behavior
- Sign - is the curve positive or negative in an interval
- Direction - is the curve increasing or decreasing in an interval as *x* increases?
- Monotonicity - is it ***always*** increasing or decreasing?
- Curvature - is the curve "concave up" or "concave down" in an interval
- Continuity - is the curve continuous
- Is it a function? Is it one-to-one?

Derivatives are very useful in characterizing properties of curves.

Increasing means:

- $f(x)$ is **increasing** on S $\Leftrightarrow$ $f(a) < f(b)$ for all $a, b \in S$ such that $a < b$.
- $f(x)$ is **decreasing** on S $\Leftrightarrow$ $f(a) > f(b)$ for all $a, b \in S$ such that $a < b$.

On an interval where *f* is ***increasing*** the slope of any tangent line is positive.

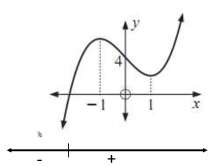
- $f(x)$ is **increasing** on S $\Leftrightarrow$ $f'(x) \geqslant 0$ for all x in S
- $f(x)$ is **decreasing** on S $\Leftrightarrow$ $f'(x) \leqslant 0$ for all x in S.

A ***monotone increasing*** function is increasing on all $x \in \mathbb{R}$

A ***monotone decreasing*** function is decreasing on all $x \in \mathbb{R}$

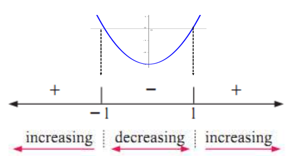
Sign diagrams are useful to display the behavior of a function.
Consider the function:

$$f(x) = x^3 - 3x + 4$$



Its sign diagram shows the intervals on *x* where the function is positive or negative.
If the function is continuous, a change in sign must happen where *f*(*x*) = 0

$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ &= 3(x^2 - 1) \\ &= 3(x+1)(x-1) \end{aligned}$$



The sign of the derivative tells whether the function is increasing (*f'*>0) or decreasing (*f'*<0) over a particular interval. The sign diagram of the derivative shows where the function is increasing or decreasing.

Consider $f(x) = \dfrac{2x-3}{x^2+2x-3}$.

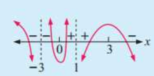
a Show that $f'(x) = \dfrac{-2x(x-3)}{(x+3)^2(x-1)^2}$ and draw its sign diagram.

b Hence, find intervals where $y = f(x)$ is increasing or decreasing.

a $f(x) = \dfrac{2x-3}{x^2+2x-3}$

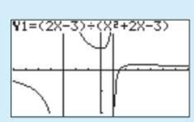
$$\begin{aligned} f'(x) &= \frac{2(x^2+2x-3) - (2x-3)(2x+2)}{(x^2+2x-3)^2} \qquad \{\text{quotient rule}\} \\ &= \frac{2x^2+4x-6-[4x^2-2x-6]}{((x-1)(x+3))^2} \\ &= \frac{-2x^2+6x}{(x-1)^2(x+3)^2} \\ &= \frac{-2x(x-3)}{(x-1)^2(x+3)^2} \end{aligned}$$

which has sign diagram



b $f(x)$ is increasing for $0 \leqslant x < 1$ and for $1 < x \leqslant 3$.

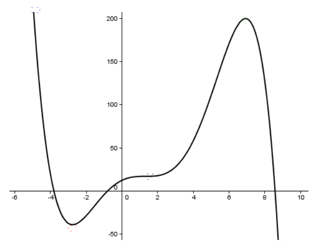
$f(x)$ is decreasing for $x < -3$ and for $-3 < x \leqslant 0$ and for $x \geqslant 3$.



A ***stationary*** point is a place where the function is ***neither increasing nor decreasing***.
Using derivatives, this means that:

> A **stationary point** of a function is a point such that $f'(x) = 0$.

Stationary points come in ***three*** flavors. To understand them, consider the following curve:



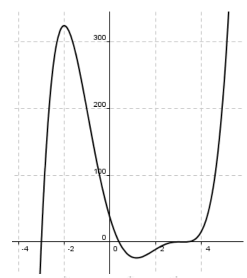
Local max: $f'(x) = 0$
Local min: $f'(x) = 0$

Notice that when analyzing extremes of functions with a restricted domain you must also consider the
Global max:
Global min:
These are ***not*** stationary points.

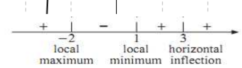
Notice that the slope can be flat and $f'(x) = 0$ at places that are ***not*** local extremes.
Such places are called ***horizontal inflection points***. (British: ***inflexion***)

Big idea: For a point to be a local extreme (min or max) $f'(x) = 0$ at the point ***and*** $f'(x)$ ***must have different signs on either side of the point***. This is one reason to draw the sign diagram.

Try one:
Draw the sign diagram of the gradient function of the graph shown

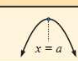
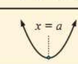
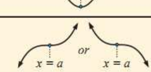


The sign diagram of its gradient function is:



Summary

A local minimum of $f(x)$ occurs at $x = a$ if $f'(a) = 0$ and $f'(x)$ changes from - to + at a
A local maximum of $f(x)$ occurs at $x = a$ if $f'(a) = 0$ and $f'(x)$ changes from + to - at a
If $f'(a) = 0$ but $f(x)$ does not change sign at $x = a$, then a is a point of horizontal inflection

Stationary point	*Sign diagram of $f'(x)$ near $x = a$*	*Shape of curve near $x = a$*
local maximum	+ | − (a)	
local minimum	− | + (a)	
horizontal inflection or stationary inflection	+ | + (a) *or* − | − (a)	

E RATIONAL FUNCTIONS

We have looked at these some but we'll revisit with better understanding of derivatives.

Rational Functions

are functions that can be written as a ratio of two polynomials.
That is $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials.

Can you describe some features of a rational function? What is its domain? Where does it cross the x-axis? What happens as x goes toward $\pm\infty$?

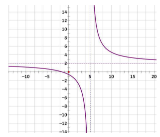
Rational functions may have:

- ***Vertical asymptotes***: at values of x where $q(x) = 0$.
- ***Zeros or roots***: at values of x where the $p(x) = 0$.
- ***Holes***: at values of x that correspond to the zeros of binomials that can be cancelled from the numerator and denominator.
- ***Horizontal asymptotes***: a y value that the function approaches as $|x| \Rightarrow \infty$

Consider the function $\frac{(2x+3)}{(x-5)}$

Vertical asymptote at ?

Horizontal asymptote at ?



Notation:
Horizontal asymptotes

as $x \to \infty$, $y \to 2$ from above or $y \to 2^+$
as $x \to -\infty$, $y \to 2$ from below or $y \to 2^-$

Vertical asymptotes

as $x \to 5$ from the left, $y \to \infty$ or as $x \to 5^-$, $y \to \infty$
as $x \to 5$ from the right, $y \to \infty$ or as $x \to 5^+$, $y \to \infty$

To work with rational function you need to understand how polynomials and algebraic fractions behave.

Consider $f(x) = \frac{3x-9}{x^2-x-2}$.

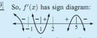
a Determine the equations of any asymptotes.
b Find $f'(x)$ and determine the position and nature of any stationary points.
c Find the axes intercepts. **d** Sketch the graph of the function.

a $f(x) = \frac{3x-9}{x^2-x-2} = \frac{3x-9}{(x-2)(x+1)}$

Vertical asymptotes are $x = 2$ and $x = -1$ {when the denominator is 0}
Horizontal asymptote is $y = 0$ {as $|x| \to \infty$, $f(x) \to 0$}

b
$$\begin{aligned} f'(x) &= \frac{3(x^2-x-2)-(3x-9)(2x-1)}{(x-2)^2(x+1)^2} \quad \text{\{Quotient rule\}} \\ &= \frac{3x^2-3x-6-[6x^2-21x+9]}{(x-2)^2(x+1)^2} \\ &= \frac{-3x^2+18x-15}{(x-2)^2(x+1)^2} \\ &= \frac{-3(x^2-6x+5)}{(x-2)^2(x+1)^2} \\ &= \frac{-3(x-5)(x-1)}{(x-2)^2(x+1)^2} \end{aligned}$$

So, $f'(x)$ has sign diagram:



$\therefore$ a local maximum when $x = 5$ and a local minimum when $x = 1$

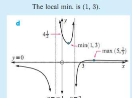
The local max. is $(5, \frac{1}{3})$.

The local min. is $(1, 3)$.

c Cuts the x-axis when $y = 0$
$\therefore$ $3x - 9 = 0$ or $x = 3$
So, the x-intercept is 3.

Cuts the y-axis when $x = 0$
$\therefore$ $y = \frac{-9}{-2} = 4\frac{1}{2}$
So, the y-intercept is $4\frac{1}{2}$.

d



For $f(x) = \frac{x^2-3x+2}{x^2+3x+2}$.

a Determine the equations of its asymptotes.
b Find $f'(x)$ and determine the position and nature of any turning points.
c Find the axes intercepts.
d Sketch the graph of the function.

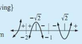
a $f(x) = \frac{x^2-3x+2}{x^2+3x+2} = \frac{1-\frac{3}{x}+\frac{2}{x^2}}{1+\frac{3}{x}+\frac{2}{x^2}}$ so as $|x| \to \infty$, $y \to 1$
$\therefore$ HA is $y = 1$

$f(x) = \frac{x^2-3x+2}{(x+1)(x+2)}$ $\therefore$ vertical asymptotes are $x = -1$ and $x = -2$

b $f'(x) = \frac{(2x-3)(x^2+3x+2)-(x^2-3x+2)(2x+3)}{(x+1)^2(x+2)^2}$

$\therefore$ $f'(x) = \frac{6x^2-12}{(x+1)^2(x+2)^2}$ {on simplifying}

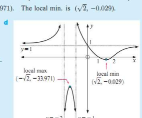
$= \frac{6(x+\sqrt{2})(x-\sqrt{2})}{(x+1)^2(x+2)^2}$ and has sign diagram



So, we have a local maximum at $x = -\sqrt{2}$ and a local minimum at $x = \sqrt{2}$.
The local max. is $(-\sqrt{2}, -33.971)$. The local min. is $(\sqrt{2}, -0.029)$.

c Cuts the x-axis when $y = 0$
$\therefore$ $x^2 - 3x + 2 = 0$
$\therefore$ $(x-1)(x-2) = 0$
$\therefore$ $x = 1$ or 2
So, the x-intercepts are 1 and 2.
Cuts the y-axis when $x = 0$
$\therefore$ $y = \frac{2}{2} = 1$
So, the y-intercept is 1.

d



For $f(x) = \frac{-x^2+4x-7}{x-1}$:

a Determine the equation of its asymptotes.
b Find $f'(x)$ and determine the position and nature of any turning points.
c Find the axes intercepts. **d** Sketch the graph of the function.

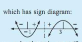
a $f(x) = \frac{-x^2+4x-7}{x-1} = -x + 3 - \frac{4}{x-1}$

$$\begin{array}{r|rrr} 1 & -1 & 4 & -7 \\ & 0 & -1 & 3 \\ \hline & -1 & 3 & -4 \end{array}$$

$\therefore$ a vertical asymptote is $x = 1$ {as $x \to 1$, $|f(x)| \to \infty$}
and an oblique asymptote is $y = -x + 3$ {as $|x| \to \infty$, $y \to -x + 3$}

b
$$\begin{aligned} f'(x) &= \frac{(-2x+4)(x-1)-(-x^2+4x-7)\times 1}{(x-1)^2} \\ &= \frac{-2x^2+6x-4+x^2-4x+7}{(x-1)^2} \\ &= \frac{-x^2+2x+3}{(x-1)^2} \\ &= \frac{-(x^2-2x-3)}{(x-1)^2} \\ &= -\frac{(x+1)(x-3)}{(x-1)^2} \end{aligned}$$

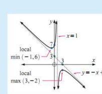
which has sign diagram:



$\therefore$ a local maximum at $(3, -2)$ and a local minimum at $(-1, 6)$

c Cuts the x-axis when $y = 0$,
$\therefore$ $-x^2 + 4x - 7 = 0$
$\therefore$ $x^2 - 4x + 7 = 0$
$\therefore$ $\Delta = 16 - 4 \times 1 \times 7 < 0$
so there are no real roots
$\therefore$ does not cut the x-axis
Cuts the y-axis when $x = 0$.
$\therefore$ $y = \frac{-7}{-1} = 7$ $\therefore$ y-intercept is 7.

d



Looks like a lot, but much is review.
Expect to spend an hour per night!
Come in if you have questions!
The simple fact is that it takes ***practice*** to absorb these ideas.

18D.1: #1ef,2jln,4,5 (Curve properties)
18D.2: #1,2efh,5,6 (Stationary points)
18E.1: #2all (Rational (linear))
18E.2: #2all (Rational (1 quadratic))
18E.3: #2all (Rational (2 quadratics))
QB #10

We'll do 18G on Fri, then 18F on Monday
Art Students - do 18G HW for Monday. The lesson in the book is sufficient.

18G: #2-16 even (more as needed) (Optimization problems)
QB #14

Present 18D.1 #2ln,4 18D.2#2e,6 18E.1.2.3 #2d

G OPTIMISATION

Perhaps the most common application of derivatives is in finding a minimum or a maximum of some function in some real world application. The process is called optimization.

A reminder:

Finding maxima and minima

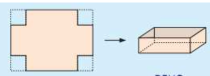
A local minimum of $f(x)$ occurs at $x = a$ if $f'(a) = 0$ and $f'(x)$ changes from - to + at a
A local maximum of $f(x)$ occurs at $x = a$ if $f'(a) = 0$ and $f'(x)$ changes from + to - at a
If $f'(a) = 0$ but $f'(x)$ does not change sign at $x = a$, then a is a point of horizontal inflection

Some hints on optimization problems:

> ***Draw a diagram!***
> ***Label it!***
> ***Define your variables and write them down***
> ***Write a conceptual block diagram of your equation if it's complex***
> ***Include units - use them to confirm your equations and results***
> ***Apply the derivative tests completely (= 0 is not enough!)***
> ***Consider the global max and min in your analysis***
> ***Answer the question!***

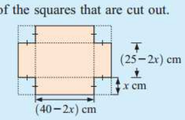
A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate, and then folding the metal to form the container.

What size squares must be cut out to produce the cake dish of maximum volume?



DEMO

Step 1: Let x cm be the side lengths of the squares that are cut out.



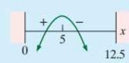
Step 2:
Volume = length × width × depth
$= (40-2x)(25-2x)x$
$= (1000-80x-50x+4x^2)x$
$= 1000x - 130x^2 + 4x^3$ cm^3

Notice that $x > 0$ and $25 - 2x > 0$
$\therefore \quad 0 < x < 12.5$

Step 3:
$$\frac{dV}{dx} = 12x^2 - 260x + 1000$$
$$= 4(3x^2 - 65x + 250)$$
$$= 4(3x-50)(x-5)$$
$$\therefore \quad \frac{dV}{dx} = 0 \quad \text{when} \quad x = \tfrac{50}{3} = 16\tfrac{2}{3} \quad \text{or} \quad x = 5$$

Step 4: **Sign diagram test**

$\frac{dV}{dx}$ has sign diagram:



or **Second derivative test**

$\frac{d^2V}{dx^2} = 24x - 260$ so when $x = 5$, $\frac{d^2V}{dx^2} = -140$ which is < 0

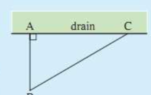
$\therefore$ the shape is $\cap$ and we have a local maximum.

So, the maximum volume is obtained when $x = 5$, which is when 5 cm squares are cut from the corners.

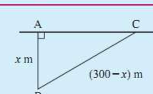
An animal enclosure is a right angled triangle with one leg being a drain. The farmer has 300 m of fencing available for the other two sides, AB and BC.

a Show that $AC = \sqrt{90\,000 - 600x}$ if $AB = x$ m.

b Find the maximum area of the triangular enclosure.
Hint: If the area is A m^2, find A^2 in terms of x.
A is a maximum when A^2 takes its maximum value.



a $(AC)^2 + x^2 = (300-x)^2$ {Pythagoras}
$\therefore \quad (AC)^2 = 90\,000 - 600x + x^2 - x^2$
$= 90\,000 - 600x$
$\therefore \quad AC = \sqrt{90\,000 - 600x}$

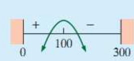


b The area of triangle ABC is
$A(x) = \frac{1}{2}(\text{base} \times \text{altitude})$
$= \frac{1}{2}(AC \times x)$
$= \frac{1}{2}x\sqrt{90\,000 - 600x}$

$0 < x < 300$

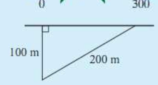
$\therefore \quad [A(x)]^2 = \frac{x^2}{4}(90\,000 - 600x) = 22\,500x^2 - 150x^3$

$\therefore \quad \frac{d}{dx}[A(x)]^2 = 45\,000x - 450x^2$
$= 450x(100 - x)$ with sign diagram:



$A(x)$ is maximised when $x = 100$

so $A_{max} = \frac{1}{2}(100)\sqrt{90\,000 - 60\,000}$
≈ 8660 m^2



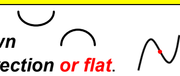
18G: #2-16 even (more as needed) (Optimization problems)
QB #14

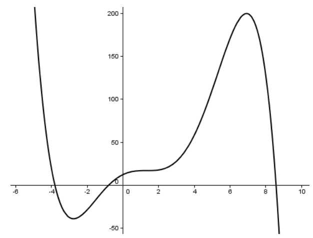
F INFLECTIONS AND SHAPE

The ***second derivative*** describes the ***curvature*** of a function.

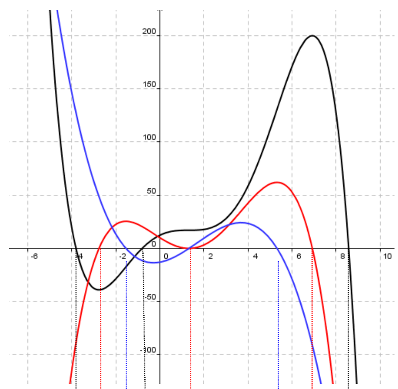
Curvature

$f''(x) > 0$: curvature of $f(x)$ is ***concave up***
$f''(x) < 0$: curvature of $f(x)$ is ***concave down***
$f''(x) = 0$: curvature of $f(x)$ is ***changing direction or flat***.





From inspection, try to sketch the first and second derivatives of the function above



Using sign diagrams to characterize a function:

$f(x)$	+	-	+	-
$f'(x)$	-	+	+	-
$f''(x)$	+	-	+	-

Using the second derivative to identify maxima and minima.

You will recall that:

Finding maxima and minima (by testing sign change of f')

A local minimum of $f(x)$ occurs at $x = a$ if $f'(a) = 0$ and $f'(x)$ changes from - to + at a
A local maximum of $f(x)$ occurs at $x = a$ if $f'(a) = 0$ and $f'(x)$ changes from + to - at a
If $f'(a) = 0$ and $f'(x)$ does not change sign at $x = a$, then a is a point of horizontal inflection

Since $f''(x)$ represents concave up or down, we can use it to determine whether a place where $f'(x) = 0$ is a minimum or a maximum. This is sometimes easier than trying to determine whether and how the first derivative changes sign. In short:

Finding maxima and minima (using f'')

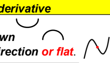
A local minimum of $f(x)$ occurs at $x = a$ if $f'(a) = 0$ and $f''(a) > 0$
A local maximum of $f(x)$ occurs at $x = a$ if $f'(a) = 0$ and $f''(a) < 0$
If $f'(a) = 0$ and $f''(a) = 0$, then a is a point of horizontal inflection

Use the method that is easiest!

Recall that:

Interpretation of second derivative

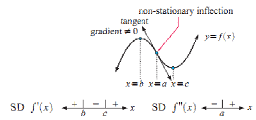
$f''(x) > 0$: curvature of $f(x)$ is ***concave up***
$f''(x) < 0$: curvature of $f(x)$ is ***concave down***
$f''(x) = 0$: curvature of $f(x)$ is ***changing direction*** ***or flat***.



Let's look more closely at the case of $f''(x) = 0$.

A ***point of inflection*** represents a change of curvature. For a to be an inflection point of f, it must be true that:

$f''(a) = 0$ ***and***
$f''(x)$ changes sign at a ($f'''(a) \neq 0$)

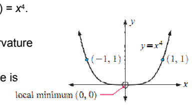


To see why the sign must change, consider $f(x) = x^4$.

$f'(x) = 4x^3$

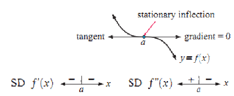
$f''(x) = 12x^2$ so $f''(0) = 0$. But clearly the curvature does not change at $x = 0$.

$f'''(x) = 24x$ so $f'''(0) = 0$. The next derivative is zero, thus this is not an inflection point



A ***stationary point of inflection*** or ***horizontal point of inflection*** represents a change of curvature at a place where the slope is zero. For a to be a stationary inflection point of f, it must be true that:

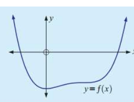
$f''(a) = 0$ ***and***
$f''(x)$ changes sign at a ($f'''(a) \neq 0$) ***and***
$f'(a) = 0$



Consider $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$.

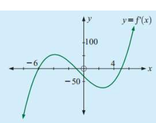
a Find and classify all points where $f'(x) = 0$.
b Find and classify all points of inflection.
c Find intervals where the function is increasing or decreasing.
d Find intervals where the function is concave up or down.
e Sketch the function showing *all* important features.

Using the graph of $y = f(x)$ alongside, sketch the graphs of $y = f'(x)$ and $y = f''(x)$.



The graph alongside shows a gradient function $y = f'(x)$.

Sketch a graph which could be $y = f(x)$, showing clearly the x-values corresponding to all stationary points and points of inflection.



18F.1: #2defgh (Inflection points)
18F.2: #1bc,2 (Curve relationships)