Calculus Overview

Unit Plan available on web page

Topic 7—Calculus

36 hrs

Aims

The aim of this section is to introduce students to the basic concepts and techniques of differential and integral calculus and their application.

Details

	Content	Amplifications/inclusions	Exclusions
7.1	Informal ideas of limit and convergence.	Only an informal treatment of limit and convergence, for example, 0.3, 0.33, 0.333, ... converges to $\frac{1}{3}$.	
	Definition of derivative as $f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$.	Use of this definition for derivatives of polynomial functions only. Other derivatives can be justified by graphical considerations using a GDC. Familiarity with both forms of notation, $\frac{dy}{dx}$ and $f'(x)$, for the first derivative.	
	Derivative of x^n $(n \in \mathbb{Q})$, $\sin x$, $\cos x$, $\tan x$, e^x and $\ln x$.		
	Derivative interpreted as gradient function and as rate of change.	Finding equations of tangents and normals. Identifying increasing and decreasing functions.	
7.2	Differentiation of a sum and a real multiple of the functions in 7.1.		
	The chain rule for composite functions.		
	The product and quotient rules.		
	The second derivative.	Familiarity with both forms of notation, $\frac{d^2y}{dx^2}$ and $f''(x)$, for the second derivative.	
7.3	Local maximum and minimum points.	Testing for maximum or minimum using change of sign of the first derivative and using sign of the second derivative.	
	Use of the first and second derivative in optimization problems.	Examples of applications: profit, area, volume.	
7.4	Indefinite integration as anti-differentiation.		
	Indefinite integral of x^n $(n \in \mathbb{Q})$, $\sin x$, $\cos x$, $\frac{1}{x}$ and e^x.	$\int \frac{1}{x} dx = \ln x + C$, $x > 0$.	
	The composites of any of these with the linear function $ax+b$.	Example: $f'(x) = \cos(2x+3) \Rightarrow f(x) = \frac{1}{2}\sin(2x+3) + C$.	
7.5	Anti-differentiation with a boundary condition to determine the constant term.	Example: if $\frac{dy}{dx} = 3x^2 + x$ and $y = 10$ when $x = 0$, then $y = x^3 + \frac{1}{2}x^2 + 10$.	
	Definite integrals.		
	Areas under curves (between the curve and the x-axis), areas between curves.	Only the form $\int_a^b y\,dx$.	$\int_a^b x\,dy$.
	Volumes of revolution.	Revolution about the x-axis only, $V = \int_a^b \pi y^2\,dx$.	Revolution about the y-axis; $V = \int_a^b \pi x^2\,dy$.
7.6	Kinematic problems involving displacement, s, velocity, v, and acceleration, a.	$v = \frac{ds}{dt}$, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$. Area under velocity–time graph represents distance.	
7.7	Graphical behaviour of functions: tangents and normals, behaviour for large $\|x\|$, horizontal and vertical asymptotes.	Both "global" and "local" behaviour.	Oblique asymptotes.
	The significance of the second derivative; distinction between maximum and minimum points.	Use of the terms "concave-up" for $f''(x) > 0$, "concave-down" for $f''(x) < 0$.	
	Points of inflexion with zero and non-zero gradients.	At a point of inflexion $f''(x) = 0$ and $f''(x)$ changes sign (concavity change). $f''(x) = 0$ is not a sufficient condition for a point of inflexion: for example, $y = x^4$ at $(0,0)$.	Points of inflexion where $f''(x)$ is not defined: for example, $y = x^{1/3}$ at $(0,0)$.

Chapter 16

Introduction to calculus

A Limits
B Finding asymptotes using limits
C Rates of change
D Calculation of areas under curves

A LIMITS

Limits - exploring a function as the variable ***approaches*** a value that might not be well defined.

> $\lim\limits_{x\to a} f(x)$ ıot the same as evaluating the function *f* at a.

> Notation and limits in the context of asymptotes.

> Evaluating limits requires special approaches and rules

Informal Definition of a Limit:

If $f(x)$ can be made as close as we like to some real number A by making x sufficiently close to a, we say that $f(x)$ approaches a **limit** of A as x approaches a, and we write

$$\lim_{x\to a} f(x) = A.$$

To find limits simplify the expression, sometimes by factoring, to eliminate any discontinuities. If there is no discontinuity at the desired value, just evaluate the expression.

Evaluate: **a** $\lim\limits_{x\to 2} x^2$ **b** $\lim\limits_{x\to 0} \dfrac{5x+x^2}{x}$

a x^2 can be made as close as we like to 4 by making x sufficiently close to 2.

$$\therefore \lim_{x\to 2} x^2 = 4.$$

b $\dfrac{5x+x^2}{x} \begin{cases} = 5+x \text{ if } x \neq 0 \\ \text{is undefined if } x = 0. \end{cases}$

$$\begin{aligned} \therefore \lim_{x\to 0} \frac{5x+x^2}{x} &= \lim_{x\to 0} \frac{x(5+x)}{x} \\ &= \lim_{x\to 0} 5+x \quad \text{since } x \neq 0 \\ &= 5 \end{aligned}$$

It can be shown that certain properties apply to limits:

- $\lim\limits_{x\to a} c = c$
- $\lim\limits_{x\to a} c\,f(x) = c \lim\limits_{x\to a} f(x)$
- $\lim\limits_{x\to a} [f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a} [f(x)g(x)] = \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a} g(x)$
- $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} f(x) \div \lim\limits_{x\to a} g(x)$ provided $\lim\limits_{x\to a} g(x) \neq 0.$

Use the rules for limits to evaluate:

a $\lim\limits_{x\to 3} (x+2)(x-1)$ **b** $\lim\limits_{x\to 1} \dfrac{x^2+2}{x-2}$

Describe the results in terms of convergence.

a As $x \to 3$, $x+2 \to 5$ and $x-1 \to 2$

$$\therefore \lim_{x\to 3} (x+2)(x-1) = 5 \times 2 = 10$$

As $x \to 3$, $(x+2)(x-1)$ *converges* to 10.

b As $x \to 1$, $x^2+2 \to 3$ and $x-2 \to -1$

$$\therefore \lim_{x\to 1} \frac{x^2+2}{x-2} = \frac{3}{-1} = -3$$

As $x \to 1$, $\dfrac{x^2+2}{x-2}$ *converges* to -3.

Indeterminant forms

$$\frac{0}{0}, \frac{0}{\pm\infty}, \frac{\pm\infty}{0}, \frac{\pm\infty}{\pm\infty}$$

Expressions that evaluate to ratios involving 0 and/or ∞ are called ***indeterminant*** and require special treatment. You must rearrange expressions to see what value the expression will approach. A common trick is to divide the numerator and denominator by some power of the variable.

Evaluate the following limits:

a $\lim\limits_{x\to\infty} \dfrac{2x+3}{x-4}$ **b** $\lim\limits_{x\to\infty} \dfrac{x^2-3x+2}{1-x^2}$

a

$$\begin{aligned} &\lim_{x\to\infty} \frac{2x+3}{x-4} \\ &= \lim_{x\to\infty} \frac{2+\frac{3}{x}}{1-\frac{4}{x}} \quad \{\text{dividing each term in both numerator and denominator by } x\} \\ &= \frac{2}{1} \quad \{\text{as } x\to\infty,\ \tfrac{3}{x}\to 0 \text{ and } \tfrac{4}{x}\to 0\} \\ &= 2 \end{aligned}$$

or

$$\begin{aligned} &\lim_{x\to\infty} \frac{2x+3}{x-4} \\ &= \lim_{x\to\infty} \frac{2(x-4)+8+3}{x-4} \\ &= \lim_{x\to\infty} \left(2+\frac{11}{x-4}\right) \\ &= 2 \quad \{\text{as } \lim_{x\to\infty} \frac{11}{x-4} = 0\} \end{aligned}$$

b

$$\begin{aligned} &\lim_{x\to\infty} \frac{x^2-3x+2}{1-x^2} \\ &= \lim_{x\to\infty} \frac{1-\frac{3}{x}+\frac{2}{x^2}}{\frac{1}{x^2}-1} \quad \{\text{dividing each term by } x^2\} \\ &= \frac{1}{-1} \quad \{\text{as } x\to\infty,\ \tfrac{3}{x}\to 0,\ \tfrac{2}{x^2}\to 0, \text{ and } \tfrac{1}{x^2}\to 0\} \\ &= -1 \end{aligned}$$

or

$$\begin{aligned} &\lim_{x\to\infty} \frac{x^2-3x+2}{1-x^2} \\ &= \lim_{x\to\infty} \frac{-x^2+3x-2}{x^2-1} \\ &= \lim_{x\to\infty} \frac{-(x^2-1)+3x-3}{x^2-1} \\ &= \lim_{x\to\infty} \left(-1+\frac{3x-3}{x^2-1}\right) \\ &= -1 \quad \{\text{as } \lim_{x\to\infty} \frac{3x-3}{x^2-1} = 0\} \end{aligned}$$

Try some:

a $\lim\limits_{x\to 0} \dfrac{x^2-3x}{x}$ **b** $\lim\limits_{h\to 0} \dfrac{2h^2+6h}{h}$ **c** $\lim\limits_{h\to 0} \dfrac{h^3-8h}{h}$

-3 6 -8

a $\lim\limits_{x\to\infty} \dfrac{1}{x}$ **b** $\lim\limits_{x\to\infty} \dfrac{3x-2}{x+1}$ **c** $\lim\limits_{x\to\infty} \dfrac{1-2x}{3x+2}$

0 3 -2/3

16A: #1fgh, 2def ,3def (Limits)

There is a more precise definition of limits but we will not go into it in this course.

Precise statement

The (ε, δ)-definition of the limit of a function is as follows[1]:

Let *f* be a function defined on an open interval containing *c* (except possibly at *c*) and let *L* be a real number. Then the formula

$$\lim_{x\to c} f(x) = L$$

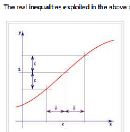
means

for each real ε > 0 there exists a real δ > 0 such that for all *x* with 0 < |*x* − *c*| < δ, we have |*f*(*x*) − *L*| < ε,

or, symbolically,

$$\forall \varepsilon > 0\ \exists\ \delta > 0 : \forall x\ (0 < |x-c| < \delta \ \Rightarrow\ |f(x) - L| < \varepsilon).$$

The real inequalities exploited in the above definition were pioneered by Bolzano and Cauchy and formalized by Weierstrass.



Whenever a point *x* is within δ units of *c*, *f*(*x*) is within ε units of *L*

B FINDING ASYMPTOTES USING LIMITS

Limits arise when looking at functions with asymptotes (often these are ***rational functions*** or functions of the form $p(x)/q(x)$ where p and q are both polynomials). Consider

$$f(x) = \frac{2x^2 - 18}{x^2 - 4x - 21}$$ To explore asymptotes, factor as much as you can and cancel terms

$$= \frac{2(x+3)(x-3)}{(x+3)(x-7)} = \frac{2(x-3)}{(x-7)}$$ Note a problem at $x = 7$.

Division by zero creates a ***vertical asymptote***.

To define it precisely we need to look at the ***sign*** of f very close to but on either side of $x = 7$.

When x is just < 7, f is big and negative (num is >0, denom is <0) so $\lim_{x \to 7^-} f(x) = -\infty$

When x is just > 7, f is positive (num is >0, denom is >0) so $\lim_{x \to 7^+} f(x) = +\infty$

Horizontal asymptotes

Let's also look at what happens as x gets very large. You may recall that:

> As x gets large, the highest power of x in a polynomial dominates the value of the function.

In our case, the numerator and denominator are both quadratic Thus, as x gets large, they both approach infinity ***at the same rate***. The lower order terms become irrelevant because:

$$f(x) = \frac{2(x-3)}{x-7} = \frac{2x-6}{x-7} = \frac{2x-14+8}{x-7} = \frac{2(x-7)+8}{x-7} = \frac{2(x-7)}{x-7} + \frac{8}{x-7} = 2 + \frac{8}{x-7}$$

Note that as $x \Rightarrow +\infty$, $f(x) \Rightarrow 2$ ***from above*** since $\frac{8}{x-7} > 0$ for $x > 0$

Likewise, as $x \Rightarrow -\infty$, $f(x) \Rightarrow 2$ ***from below*** since $\frac{8}{x-7} < 0$ for $x < 0$

What do you think this would give?

$$\lim_{x \to \pm\infty} \frac{3x^3 - 2x + 5}{2x^3 + 4x^2 - 8x + 9} = \frac{3}{2}$$

These two limits (+ and -) define a horizontal asymptote of the function.

Horizontal asymptotes in rational functions occur as $x \Rightarrow \pm\infty$ when the degree of the numerator and denominator are equal. The H.A. is at y = "the ratio of the two leading coefficients"

Summary of Behavior of Rational Functions

1. When the degree of the numerator is ***greater than*** the degree of the denominator by more than one, the function will diverge as x approaches $\pm\infty$.
2. When the degree of the numerator is ***equal to*** than the degree of the denominator, the function will approach a ***horizontal asymptote*** as x approaches $\pm\infty$. The y-value of the asymptote is the ratio of the coefficients of the leading terms.
3. When the degree of the numerator is ***less than*** the degree of the denominator, the function will approach a ***horizontal asymptote at y = 0*** as x approaches $\pm\infty$.

Don't memorize, ***understand why***.

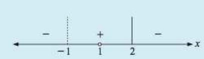
Try one:

Find any asymptotes of the function $f : x \to \frac{x^2 - 3x + 2}{1 - x^2}$ and discuss the behaviour of $f(x)$ near these asymptotes.

We notice that $f(x) = \frac{(x-2)\cancel{(x-1)}^{-1}}{\cancel{(1-x)}(1+x)} = \frac{-(x-2)}{1+x}$ provided $x \neq 1$.

So, there is a point discontinuity at $x = 1$.
Also, when $x = -1$, $f(x)$ is undefined. {dividing by zero}
This indicates that $x = -1$ is a vertical asymptote.

The sign diagram for $f(x)$ is:



As $x \to -1$ (left), $f(x) \to -\infty$.
As $x \to -1$ (right), $f(x) \to +\infty$.
$\therefore$ $x = -1$ is a VA.

For $x \neq 1$, $f(x) = \frac{-(x-2)}{1+x} = \frac{-x+2}{x+1} = \frac{-(x+1)+3}{x+1} = -1 + \frac{3}{x+1}$

As $x \to \infty$, $f(x) \to -1$ (above) as $\frac{3}{x+1} \to 0$ and is > 0.

As $x \to -\infty$, $f(x) \to -1$ (below) as $\frac{3}{x+1} \to 0$ and is < 0.

$\therefore$ HA is $y = -1$ (see **Example 3** part b)

Do the investigation in section 16 C in class. What did you conclude?

The ***instantaneous rate of change*** of a function $f(x)$ with respect to x at a point x = a is given by the ***gradient*** (***slope***) of the line ***tangent*** to the function at x = a. This is also known as the ***gradient of the curve*** at x = a.

16B: #1 (Limits as asymptotes)

Chapter 17 Differential calculus

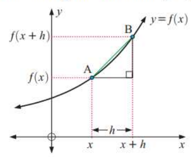
A The derivative function
B Derivatives at a given x-value
C Simple rules of differentiation
D The chain rule
E The product rule
F The quotient rule
G Tangents and normals
H The second derivative

A THE DERIVATIVE FUNCTION

Geogebra Demo: Derivatives

Summary of a Derivative

Consider a general function $y = f(x)$ where A is $(x, f(x))$ and B is $(x+h, f(x+h))$.



The chord [AB] has gradient $= \dfrac{f(x+h)-f(x)}{x+h-x}$

$= \dfrac{f(x+h)-f(x)}{h}$.

If we now let B approach A, then the gradient of [AB] approaches the gradient of the tangent at A.

So, the gradient of the tangent at the variable point $(x, f(x))$ is the limiting value of

$\dfrac{f(x+h)-f(x)}{h}$ as h approaches 0, or $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

The **gradient function**, also known as the **derived function** or **derivative function** or simply the **derivative** is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

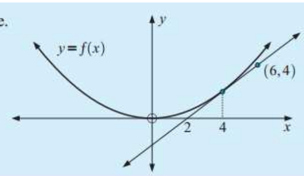
We read the derivative function as 'eff dashed x'.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is called the ***limit quotient***. Finding the derivative of a function by evaluating the limit quotient is called using ***first principles.***

Consider the graph alongside.

Find $f(4)$ and $f'(4)$.



The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 4$.

The gradient of this tangent is $f'(4)$.

The tangent passes through $(2, 0)$ and $(6, 4)$, so $f'(4) = \dfrac{4-0}{6-2} = 1$.

The equation of the tangent is $\dfrac{y-0}{x-2} = 1$

$\therefore \quad y = x - 2$

When $x = 4$, $y = 2$, so the point of contact is $(4, 2)$

$\therefore \quad f(4) = 2$ So, $f(4) = 2$ and $f'(4) = 1$.

Use the definition of $f'(x)$ to find the gradient function of $f(x) = x^2$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} \\
&= \lim_{h\to 0} (2x+h) \qquad \{\text{as } h \neq 0\} \\
&= 2x
\end{aligned}
$$

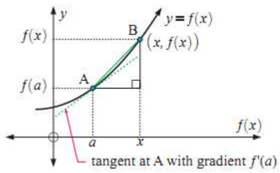
B DERIVATIVES AT A GIVEN x-VALUE

You may not need a general function that describes the derivative at any point. It may be enough to find the gradient at a single point.

One way is to evaluate the limit quotient at the given point, call it *a*. Thus, the gradient of a function *f* at a point *a* is given by:

$$\boxed{f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}}$$

Alternatively, one can imagine a variable point *x* to the right of *a* and evaluate a limit quotient that's written slightly differently:



$$\boxed{f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$$

Find, from first principles, the gradient of the tangent to $y=2x^2+3$ at the point where $x=2$.

Let $f(x)=2x^2+3$

$\therefore\ f(2)=2(2)^2+3=11$

and $f'(2)=\lim_{x\to 2}\dfrac{f(x)-f(2)}{x-2}$

$$\begin{aligned}\therefore\ f'(2)&=\lim_{x\to 2}\frac{2x^2+3-11}{x-2}\\&=\lim_{x\to 2}\frac{2x^2-8}{x-2}\\&=\lim_{x\to 2}\frac{2(x+2)\cancel{(x-2)}}{\cancel{x-2}_{\,1}}\quad\{\text{as } x\neq 2\}\\&=2\times 4\\&=8\end{aligned}$$

Confirm this on your calculator: nDeriv(function, variable, value)

nDeriv($2x^2+3$,x, 2) = 8

Use the first principles formula $f'(a)=\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ to find the instantaneous rate of change in $f(x)=x^2+2x$ at the point where $x=5$.

$f(5)=5^2+2(5)=35$

$$\begin{aligned}\text{So,}\quad f'(5)&=\lim_{h\to 0}\frac{f(5+h)-f(5)}{h}\\&=\lim_{h\to 0}\frac{(5+h)^2+2(5+h)-35}{h}\\&=\lim_{h\to 0}\frac{\cancel{25}+10h+h^2+\cancel{10}+2h-\cancel{35}}{h}\\&=\lim_{h\to 0}\frac{h^2+12h}{h}\\&=\lim_{h\to 0}\frac{\cancel{h}(h+12)}{\cancel{h}_{\,1}}\quad\{\text{as } h\neq 0\}\\&=12\end{aligned}$$

$\therefore$ the instantaneous rate of change in $f(x)$ at $x=5$ is 12.

Try it using the formula $f'(a)=\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}$

Do you have a preference?

17A.1: #1-3 (The derivative)
17A.2: #1-3 (First principles)
17B: #1cd,#2 (Derivatives from first principles)

Present 17A.2 #1,2,3e,f

C SIMPLE RULES OF DIFFERENTIATION

The result of 17A.2 #2 is fundamental and is very useful. Let's prove it:

$$f(x)=x^n$$
$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$
$$=\lim_{h\to 0}\frac{x^n+\binom{n}{1}x^{n-1}h+\binom{n}{2}x^{n-2}h^2+\ldots\binom{n}{n-1}xh^{n-1}+h^n-x^n}{h}$$
$$=\lim_{h\to 0}\frac{\binom{n}{1}x^{n-1}h+\binom{n}{2}x^{n-2}h^2+\ldots\binom{n}{n-1}xh^{n-1}+h^n}{h}$$
$$=\lim_{h\to 0}\binom{n}{1}x^{n-1}+\binom{n}{2}x^{n-2}h+\ldots\binom{n}{n-1}xh^{n-1}+h^{n-1}$$
$$=\binom{n}{1}x^{n-1}=n\cdot x^{n-1}$$

Derivative of a power function

$$f'(x^n)=nx^{(n-1)}$$

We will explore many of the properties of differentiation using a power function to practice.

Here are the first set of rules. Can you prove them from first principles?

$f(x)$	$f'(x)$	*Name of rule*
c (a constant)	0	**differentiating a constant**
x^n	nx^{n-1}	**differentiating x^n**
$c\,u(x)$	$c\,u'(x)$	**constant times a function**
$u(x)+v(x)$	$u'(x)+v'(x)$	**addition rule**

This is also a good time to discuss notation. The derivative of *y* with respect to *x* (meaning that *y* is the dependent variable and *x*is the independent variable) is also given by: $\frac{dy}{dx}$

This notation was developed by Gottfried Leibnitz around 168. He was credited with discovering infinitesimal calculus simultaneously and independently from Isaac Newton, who developed and used the "prime" notation.

The following instructions are all different versions of saying the same thing:

Find $f'(x)$

Find $\frac{dy}{dx}$

Differentiate with respect to x:

Find the gradient of the tangent to:

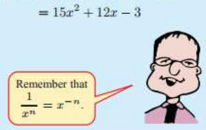
Find the gradient function of $f(x)$

Try some:

Find $f'(x)$ for $f(x)$ equal to: **a** $5x^3+6x^2-3x+2$ **b** $7x-\frac{4}{x}+\frac{3}{x^3}$

a $f(x)=5x^3+6x^2-3x+2$
$\therefore\ f'(x)=5(3x^2)+6(2x)-3(1)$
$=15x^2+12x-3$

b $f(x)=7x-\frac{4}{x}+\frac{3}{x^3}$
$=7x-4x^{-1}+3x^{-3}$
$\therefore\ f'(x)=7(1)-4(-1x^{-2})+3(-3x^{-4})$
$=7+4x^{-2}-9x^{-4}$
$=7+\frac{4}{x^2}-\frac{9}{x^4}$

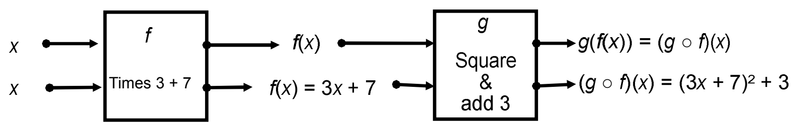


Find the slope of the line tangent to the curve $f(x)=\frac{-3}{\sqrt[4]{x^3}}$ at *x* = 2

17C: #1aeim,2all,4ace,6dh,7 (Differentiating power rule)

D THE CHAIN RULE

We begin this section by reviewing ***composite functions***



The composite of two functions is created by using the output of one function as the input to the other function. Some properties:

$(f \circ g)(x)$ is ***not*** the same as $(g \circ f)(x)$ in general

The range of the first function in a composition is the domain of the second.

Try: Given $f(x) = x^2 + 7$ and $g(x) = 2x + 4$ find $(f \circ g)(x)$ and $(g \circ f)(x)$

$(f \circ g)(x) = (2x + 4)^2 + 7$ $\quad$ $(g \circ f)(x) = 2(x^2 + 7) + 4$

Consider the function x^2 whose derivative is $2x$. What is the derivative of $(2x)^2$?

Now try differentiating $(2x + 3)^2$. Do it with and without expanding first.

Finally, try differentiating $(x^2 + 3x + 4)^2$ by guesswork and then by expanding it out.

From these short examples we can see the ***chain rule*** at work.

The Chain Rule

If $f(x) = g(h(x)) = (g \circ f)(x)$ then
$f'(x) = g'(h(x)) \cdot h'(x)$

Not recognizing the need to use the chain rule is probably the single most common source of errors in differentiation!

Although the concept is simple, it can get complex.

Find $\frac{dy}{dx}$ if: **a** $y = (x^2 - 2x)^4$ **b** $y = \frac{4}{\sqrt{1-2x}}$

a $y = (x^2 - 2x)^4$

$\therefore\ y = u^4$ where $u = x^2 - 2x$

Now $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ {chain rule}

$= 4u^3(2x - 2)$

$= 4(x^2 - 2x)^3(2x - 2)$

b $y = \frac{4}{\sqrt{1-2x}}$

$\therefore\ y = 4u^{-\frac{1}{2}}$ where $u = 1 - 2x$

Now $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ {chain rule}

$= 4 \times (-\frac{1}{2}u^{-\frac{3}{2}}) \times (-2)$

$= 4u^{-\frac{3}{2}}$

$= 4(1 - 2x)^{-\frac{3}{2}}$

What about $f(x) = \left(\frac{4}{\sqrt{2x^2 - 1}}\right)^3$

17D.1: #1,2 (Chain Rule (composite functions))
17D.2: #1ad,2adg,3ace,4 (Chain Rule)
QB: 24a,33a,40*a-c,44a-c,46a(power)

E THE PRODUCT RULE

What happens when you take the derivative of a product? Let's look at this from first principles:

$$
\begin{aligned}
\left(f(x)\cdot g(x)\right)' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)+f(x+h)g(x)-f(x+h)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)+f(x+h)g(x)-f(x+h)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)\left(g(x+h)-g(x)\right)+g(x)\left(f(x+h)-f(x)\right)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)\left(g(x+h)-g(x)\right)}{h}+\lim_{h\to 0}\frac{g(x)\left(f(x+h)-f(x)\right)}{h}\\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= f(x)g'(x)+g(x)f'(x)
\end{aligned}
$$

There are lots of notations for this - all boil down to the same thing! Another proof using Leibniz notation is given in the book.

The Product Rule

If $u(x)$ and $v(x)$ are two functions of x and $y = uv$ then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} \quad \text{or} \quad y' = u'(x)v(x) + u(x)v'(x).$$

Try a couple: (Don't forget the Chain Rule)

Find $\frac{dy}{dx}$ if: **a** $y = \sqrt{x}(2x+1)^3$ **b** $y = x^2(x^2-2x)^4$

a $y = \sqrt{x}(2x+1)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (2x+1)^3$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(2x+1)^2 \times 2$
$= 6(2x+1)^2$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$
\begin{aligned}
&= \tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + x^{\frac{1}{2}} \times 6(2x+1)^2\\
&= \tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + 6x^{\frac{1}{2}}(2x+1)^2
\end{aligned}
$$

b $y = x^2(x^2-2x)^4$ is the product of $u = x^2$ and $v = (x^2-2x)^4$

$\therefore\ u' = 2x$ and $v' = 4(x^2-2x)^3(2x-2)$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$
\begin{aligned}
&= 2x(x^2-2x)^4 + x^2 \times 4(x^2-2x)^3(2x-2)\\
&= 2x(x^2-2x)^4 + 4x^2(x^2-2x)^3(2x-2)
\end{aligned}
$$

F THE QUOTIENT RULE

Consider a function $f(x)=\dfrac{u(x)}{v(x)}=u(x)\bullet\dfrac{1}{v(x)}$

Making use of the product rule, derive a formula for $f'(x)$ in terms of u, u', v, & v'

$$f(x)=\frac{u(x)}{v(x)}=u(x)\bullet\frac{1}{v(x)}$$

$$f'(x)=u\frac{d}{dx}\left(\frac{1}{v}\right)+\frac{d}{dx}(u)\frac{1}{v}$$

$$=u\left(-v^{-2}\right)v'+u'\frac{1}{v}$$

$$=\frac{-uv'}{v^2}+\frac{u'}{v}=\frac{u'v-uv'}{v^2}$$

The Quotient Rule

So, if $Q(x)=\dfrac{u(x)}{v(x)}$ then $Q'(x)=\dfrac{u'(x)v(x)-u(x)v'(x)}{[v(x)]^2}$

or if $y=\dfrac{u}{v}$ where u and v are functions of x then $\dfrac{dy}{dx}=\dfrac{u'v-uv'}{v^2}$.

Try a couple:

Use the quotient rule to find $\dfrac{dy}{dx}$ if: **a** $y=\dfrac{1+3x}{x^2+1}$ **b** $y=\dfrac{\sqrt{x}}{(1-2x)^2}$

a $y=\dfrac{1+3x}{x^2+1}$ is a quotient with $u=1+3x$ and $v=x^2+1$

$\therefore$ $u'=3$ and $v'=2x$

Now $$\frac{dy}{dx}=\frac{u'v-uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$=\frac{3(x^2+1)-(1+3x)2x}{(x^2+1)^2}$$

$$=\frac{3x^2+3-2x-6x^2}{(x^2+1)^2}$$

$$=\frac{3-2x-3x^2}{(x^2+1)^2}$$

b $y=\dfrac{\sqrt{x}}{(1-2x)^2}$ is a quotient where $u=x^{\frac{1}{2}}$ and $v=(1-2x)^2$

$\therefore$ $u'=\frac{1}{2}x^{-\frac{1}{2}}$ and $v'=2(1-2x)^1\times(-2)$

$=-4(1-2x)$

Now $$\frac{dy}{dx}=\frac{u'v-uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$=\frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2-x^{\frac{1}{2}}\times(-4(1-2x))}{(1-2x)^4}$$

$$=\frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2+4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}$$

$$=\frac{\cancel{(1-2x)}\left[\frac{1-2x}{2\sqrt{x}}+4\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^{\cancel{4}\,3}} \quad \{\text{look for common factors}\}$$

$$=\frac{1-2x+8x}{2\sqrt{x}(1-2x)^3}$$

$$=\frac{6x+1}{2\sqrt{x}(1-2x)^3}$$

17E: #1ad,2bd,3 (Product Rule)
17F: #1cf,2bd,3,4 (Quotient Rule)

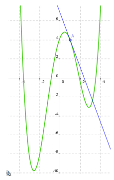
This is a ***minimum*** set of problems to do. Do more until you are proficient.

Present 17E #1d,2d 17F #1f,2d,4

G TANGENTS AND NORMALS

Recognizing that a derivative is the gradient of a curve at a point, one can use differential calculus to explore problems involving lines that are tangent and normal to curves.

Consider the diagram:



If we find the derivative at some point, $A(a, f(a))$, we have the slope and a point. Remember point-slope form of a line? $y - y_1 = m(x - x_1)$ where (x_1, y_1) is a point on a line with slope m. So we can find the equation of the line tangent to the curve:

Equation of a Line Tangent to f at x = a

$$y - f(a) = f'(a)(x - a)$$

Similarly, the equation for a line ***normal*** to a curve at a point $A(a, f(a))$ is given by:

Equation of a Line Normal to f at x = a

$$y - f(a) = -\frac{1}{f'(x)}(x - a)$$

Find the equation of the tangent to $y = \sqrt{10 - 3x}$ at the point where $x = 3$.

Let $f(x) = (10 - 3x)^{\frac{1}{2}}$

$\therefore\ f'(x) = \frac{1}{2}(10 - 3x)^{-\frac{1}{2}} \times (-3)$

$\therefore\ m_T = f'(3) = \frac{1}{2}(1)^{-\frac{1}{2}} \times (-3) = -\frac{3}{2}$

When $x = 3$, $y = \sqrt{10 - 9} = 1$

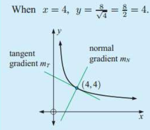
$\therefore$ the point of contact is $(3, 1)$.

So, the tangent has equation $\dfrac{y - 1}{x - 3} = -\dfrac{3}{2}$ which is $2y - 2 = -3x + 9$

or $3x + 2y = 11$

Find the equation of the normal to $y = \dfrac{8}{\sqrt{x}}$ at the point where $x = 4$.

When $x = 4$, $y = \frac{8}{\sqrt{4}} = \frac{8}{2} = 4$. So, the point of contact is $(4, 4)$.



Now as $y = 8x^{-\frac{1}{2}}$, $\dfrac{dy}{dx} = -4x^{-\frac{3}{2}}$

$\therefore$ when $x = 4$, $m_T = -4 \times 4^{-\frac{3}{2}} = -\frac{1}{2}$

$\therefore$ the normal at $(4, 4)$ has gradient $m_N = \frac{2}{1}$.

$\therefore$ the equation of the normal is

$2x - 1y = 2(4) - 1(4)$ or $2x - y = 4$.

Some more thoughtful applications...

Find the equations of any horizontal tangents to $y = x^3 - 12x + 2$.

Since $y = x^3 - 12x + 2$, $\dfrac{dy}{dx} = 3x^2 - 12$

Horizontal tangents have gradient 0, so $3x^2 - 12 = 0$

$\therefore\ 3(x^2 - 4) = 0$

$\therefore\ 3(x + 2)(x - 2) = 0$

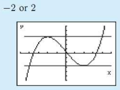
$\therefore\ x = -2$ or 2

When $x = 2$, $y = 8 - 24 + 2 = -14$

When $x = -2$, $y = -8 + 24 + 2 = 18$

$\therefore$ the points of contact are $(2, -14)$ and $(-2, 18)$

$\therefore$ the tangents are $y = -14$ and $y = 18$.



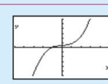
What does it mean for a tangent line to be horizontal? More on this later.

Find the coordinates of the point(s) where the tangent to $y = x^3 + x + 2$ at $(1, 4)$ meets the curve again.

Let $f(x) = x^3 + x + 2$

$\therefore\ f'(x) = 3x^2 + 1$

$\therefore\ m_T = f'(1) = 3 + 1 = 4$



$\therefore$ the tangent at $(1, 4)$ has gradient 4

and its equation is $4x - y = 4(1) - 4$ or $y = 4x$.

Now $y = 4x$ meets $y = x^3 + x + 2$ where $x^3 + x + 2 = 4x$

$\therefore\ x^3 - 3x + 2 = 0$

Since the tangent *touches* the curve when $x = 1$, $x = 1$ must be a repeated zero of $x^3 - 3x + 2$.

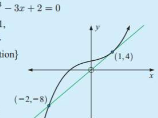
$\therefore\ (x - 1)^2(x + 2) = 0$ {by inspection}

$x^2 \times x = x^3$ $(-1)^2 \times 2 = 2$

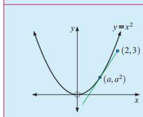
$\therefore\ x = 1$ or -2

When $x = -2$, $y = (-2)^3 + (-2) + 2 = -8$

$\therefore$ the tangent meets the curve again at $(-2, -8)$.



Find the equations of the tangents to $y = x^2$ from the external point $(2, 3)$.



Let (a, a^2) lie on $f(x) = x^2$.

Now $f'(x) = 2x$, so $f'(a) = 2a$

$\therefore$ at (a, a^2) the gradient of the tangent is $\dfrac{2a}{1}$

$\therefore$ its equation is $\dfrac{y - a^2}{x - a} = 2a$

or $y - a^2 = 2ax - 2a^2$

or $y = 2ax - a^2$

But this tangent passes through $(2, 3)$.

$\therefore\ 3 = 2a(2) - a^2$

$\therefore\ a^2 - 4a + 3 = 0$

$\therefore\ (a - 1)(a - 3) = 0$

$\therefore\ a = 1$ or 3

If $a = 1$, the tangent has equation $y = 2x - 1$ with point of contact $(1, 1)$.

If $a = 3$, the tangent has equation $y = 6x - 9$ with point of contact $(3, 9)$.

H THE SECOND DERIVATIVE

Consider the height of a ball thrown up from the top of a 60 foot building at an initial speed of 12 feet per second. The ball's height can be modeled by:

$$h(t) = -16t^2 + 12t + 60$$

The change of position over time is also known as ***velocity***. Velocity $= \dfrac{\Delta\text{Position}}{\Delta\text{Time}}$

Using derivatives, we can find instantaneous velocity by letting ΔTime $\to 0$

$$\text{Instantaneous Velocity} = \lim_{\Delta Time \to 0} \frac{\Delta Position}{\Delta Time} = \frac{d(\text{Position})}{d(\text{Time})} = \frac{dh}{dt} = h'(t)$$

In the example, velocity is given by $v = -32t + 12$. Notice that when $t = 0$, the velocity is 12 ft/sec (the initial speed!) and that the velocity increases (in a downward direction) 32 ft/sec every second!

That ***change in velocity with respect to time*** is called ***acceleration***. It is also the ***second derivative*** of position (in this case height) with respect to time.

$$\boxed{\text{Instantaneous Acceleration} = \lim_{\Delta Time \to 0} \frac{\Delta Velocity}{\Delta Time} = \frac{d(\text{Velocity})}{d(\text{Time})} = \frac{d^2h}{dt^2} = h''(t)}$$

In the above example, the acceleration is simply, -32 ft/sec^2 - gravity!

Notice the positions of the superscripts in Leibnitz notation - the reasoning for this will become apparent later. It is read "dee-two h dee-tee squared"

Abstractly, $f''(x)$ or $f^{(2)}(x)$ is the ***second derivative*** of f with respect to x. It represents the the "slope of the slope" or the ***curvature*** of f .

To find a second derivative, simply take the derivative of the derivative. And yes, Martha, you can continue this process (though we won't go into it in SL).

We will talk a lot more about the meaning of second derivatives in the next chapter.

17G: #3,4,5bd,6bd,7,8a,9a,10 (Tangent and Normal lines)
17H: #1bdf,2cf,3,4 (Second derivatives)
QB: 2*(a-c),6,13,16*(a-c),18*

Note: The HW for 17G and the QB are ***core calculus ideas***. Do this thoroughly! Everyone will need to present on Thursday.
It jumps right into some more complex applications of the ideas. To dip your feet into the water first, try a couple from #1 & 2.
Add **8a, 9a, and 10** to your unit plan sheet! See the website for an updated unit plan.