A LIMITS

Limits - exploring a function as the variable approaches a value that might not be well defined.

> $\lim_{x\to a} f(x)$ is not the same as evaluating the function *f* at a. 16A

> Notation and limits in the context of asymptotes 16B

> Evaluating limits requires special approaches and rules

- $\lim_{x\to a} c = c$
- $\lim_{x\to a} c\,f(x) = c \lim_{x\to a} f(x)$
- $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
- $\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
- $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) \div \lim_{x\to a} g(x)$ provided $\lim_{x\to a} g(x) \neq 0$.

a $\lim_{x\to 0} \frac{x^2-3x}{x}$ -3 **b** $\lim_{h\to 0} \frac{2h^2+6h}{h}$ 6 **c** $\lim_{h\to 0} \frac{h^3-8h}{h}$ -8

a $\lim_{x\to\infty} \frac{1}{x}$ 0 **b** $\lim_{x\to\infty} \frac{3x-2}{x+1}$ 3 **c** $\lim_{x\to\infty} \frac{1-2x}{3x+2}$ -2/3

Limits arise when looking at functions with asymptotes (often these are ***rational functions***). Consider

$f(x) = \frac{2x^2-18}{x^2-4x-21}$ To explore asymptotes, factor as much as you can and cancel terms

$= \frac{2(x+3)(x-3)}{(x+3)(x-7)} = \frac{2(x-3)}{(x-7)}$ Note a problem at $x = 7$.

Division by zero creates a ***vertical asymptote***.

To define it precisely we need to look at the ***sign*** of *f* very close to but on either side of $x = 7$.

When x is just < 7, *f* is big and negative (num is >0, denom is <0) so $\lim_{x\to 7^-} f(x) = -\infty$

When x is just > 7, *f* is positive (num is >0, denom is >0) so $\lim_{x\to 7^+} f(x) = +\infty$

Let's also look at what happens as *x* gets very large. You may recall that:

> As *x* gets large, the highest power of *x* in a polynomial dominates the value of the function

In our case, the numerator and denominator are both linear. Thus, as *x* gets large, they both approach infinity linearly and ***at the same rate***. The - 3 and the -7 become irrelevant and we have essentially

$\lim_{x\to+\infty} f(x) = \frac{2\cdot+\infty}{+\infty} = 2$ and similarly $\lim_{x\to-\infty} f(x) = \frac{2\cdot-\infty}{-\infty} = 2$

These two limits define a horizontal asymptote of the function.

Horizontal asymptotes in rational functions(may) occur as *x* approaches plus or minus infinity

Two other things to note while we're here.

The ***zeros*** of a rational function are the values of *x* that make the numerator zero (assuming they do not cause an issue in the denominator).

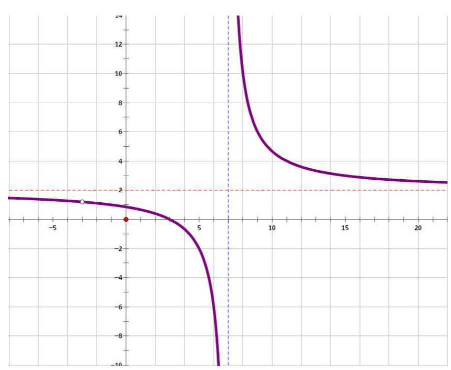
In our case, $x = 3$ is a zero.

Secondly: Notice that $x = -3$ is ***not*** a zero. Even though it makes the numerator zero, there is also an $x + 3$ in the denominator. So there is a problem in the *original function* at $x = -3$ since it would result in 0/0. It's called a ***hole*** at $x = -3$. Notice that ***after*** we cancelled the $(x + 3)$, the problem is not apparent! $f(x)$ is not defined at $x = -3$. But we can say that:

$$\lim_{x\to-3} f(x) = \lim_{x\to-3} \frac{2(x+3)(x-3)}{(x+3)(x-7)} = \frac{(2)(-6)}{-10} = \frac{6}{5}$$

It appears that we just plugged -3 into the reduced form which works in this case (but not always...)

We'll finish by looking at the graph of the function:



A THE DERIVATIVE FUNCTION

Derivative - a function ***derived*** from another function that describes the original function's rate of change (aka *slope*, *gradient*) at any value of *x*.

> What is the instantaneous rate of change (velocity) of a falling object *t* seconds after it is dropped? 16C

> Approximations of slope using limits lead to $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ 17A

> Derive derivative functions for powers, exponentials, logs, sin, cos, tan and 1/*x*.

Powers -	17C
Exponentials -	19A
Ln & 1/x -	19C
sin, cos, tan -	20A

> Properties of derivatives

c, c*f*(*x*) and *f*(*x*) + *g*(*x*)	
Chain rule	17D
Product rule	17E
Quotient rule	17F

> Second derivatives & curvature	17H
> Applications	
Tangents and Normals	17G
Rate of change and others	Ch 18

SLCalcSketches.gsp

IB uses the word ***gradient*** interchangeably with ***slope***. The ***gradient function*** is the ***derivative***.

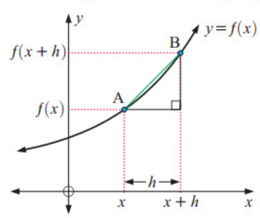
Newton's notation was *f'*(*x*)

Leibniz used $\frac{dy}{dx}$ which clarifies the variable that we are deriving ***with respect to***.

Heavyside used $D_x f$ and you may also see Df

Summary of a Derivative

Consider a general function $y = f(x)$ where A is $(x, f(x))$ and B is $(x+h, f(x+h))$.



The chord [AB] has gradient $= \frac{f(x+h) - f(x)}{x+h-x}$

$= \frac{f(x+h) - f(x)}{h}$.

If we now let B approach A, then the gradient of [AB] approaches the gradient of the tangent at A.

So, the gradient of the tangent at the variable point $(x, f(x))$ is the limiting value of

$\frac{f(x+h) - f(x)}{h}$ as h approaches 0, or $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Derivatives of key functions

We can develop the derivative functions for various common functions ***from first principles*** by evaluating limits. Once we have done this, we memorize the result for the derivatives of these common functions. For SL, you need to know the derivatives of the following functions:

Powers -	17C
Exponentials -	19A
Natural log -	19C
sin, cos, tan -	20A

Let's develop the derivative of power functions from first principles to remind you of that approach:

Given $f(x) = x^n$

We want to find $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

Plug in the definition of *f* $= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

Use the binomial expansion to rewrite the numerator $= \lim_{h\to 0} \frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots \binom{n}{n-1}xh^{n-1} + h^n - x^n}{h}$

The first and last term cancel $= \lim_{h\to 0} \frac{\binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots \binom{n}{n-1}xh^{n-1} + h^n}{h}$

Cancel the common factor *h* from top and bottom $= \lim_{h\to 0} \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \dots \binom{n}{n-1}xh^{n-1} + h^{n-1}$

Now we can take the limit with no problem. Since *n* choose 1 is just *n* we have the result $= \binom{n}{1}x^{n-1} = n \cdot x^{n-1}$

Derivatives of key functions

Name	$f(x)$	$f'(x)$ or $\frac{dy}{dx}$
Constant Function	$f(x) = c$	$f'(x) = 0$
Power Function	$f(x) = x^n$	$f'(x) = nx^{n-1}$
General Exponential Function	$f(x) = a^x$	$f'(x) = ka^x$ where $k = f'(0)$
Natural Exponential Function	$f(x) = e^x$	$f'(x) = e^x$
Natural Log Function	$f(x) = \ln x$	$f'(x) = \frac{1}{x}$
Sine Function	$f(x) = \sin x$	$f'(x) = \cos x$
Cosine Function	$f(x) = \cos x$	$f'(x) = -\sin x$
Tangent Function	$f(x) = \tan x$	$f'(x) = \frac{1}{\cos^2 x}$

These are all on the formula sheet - don't sweat it

Differentiation rules

How do we take the derivative of a combination of functions?
First, consider the various ways to combine functions:

Finding a derivative is also called ***differentiating***.

Add or subtract	$u(x) + v(x)$ or $u(x) - v(x)$
Multiply	$u(x)\cdot v(x)$
Divide	$\frac{u(x)}{v(x)}$
Compose	$(u \circ v)(x)$ or $u(v(x))$

A summary. These are proven from first principles in the book. It might be good to look through a couple.

Add or subtract	$f(x) = u(x) \pm v(x)$	$f'(x) = u'(x) \pm v'(x)$	*Addition Rule*
Multiply	$f(x) = u(x)\cdot v(x)$	$f'(x) = u'(x)\cdot v(x) + u(x)\cdot v'(x)$	*Product Rule*
Divide	$f(x) = \frac{u(x)}{v(x)}$	$f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{v^2(x)}$	*Quotient Rule*
Compose	$f(x) = (u \circ v)(x)$ or $u(v(x))$	$f'(x) = u'(v(x))\cdot v'(x)$	*Chain Rule*

Here they are in another form. Notice the Leibniz notation for the Chain Rule below.
Again, these are all in your Formula Sheet. Print it out and get to know it!

Function	*Derivative*	*Name*
c, a constant	0	
$mx + c$, m and c are constants	m	
x^n	nx^{n-1}	**power rule**
$cu(x)$	$cu'(x)$	
$u(x) + v(x)$	$u'(x) + v'(x)$	**addition rule**
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	**product rule**
$\frac{u(x)}{v(x)}$	$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$	**quotient rule**
$y = f(u)$ where $u = u(x)$	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	**chain rule**
e^x	e^x	
$e^{f(x)}$	$e^{f(x)} f'(x)$	
$\ln x$	$\frac{1}{x}$	
$\ln f(x)$	$\frac{f'(x)}{f(x)}$	
$[f(x)]^n$	$n[f(x)]^{n-1} f'(x)$	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\frac{1}{\cos^2 x}$	

Second derivatives

If one takes the derivative of a derivative, we are finding the ... rate of change of ... the rate of change of the original function.

The ***second derivative*** may be written as $f''(x)$ $\frac{d^2y}{dx^2}$ $D_x^2 f$ $D^2 f$

Graphically, the second derivative represents the ***curvature*** of a function.
- f'' > 0 represents concave up
- f'' < 0 represents concave down
- f'' = 0 represents a ***point of inflection*** where the curvature ***may*** change

Physically, the second derivative represents the ***acceleration*** of something changing.
- f'' > 0 represents increasing rate of change (often velocity)
- f'' < 0 represents decreasing rate of change (often velocity)
- f'' = 0 represents constant rate of change (often velocity)

Nothing special algebraically - just write neatly and keep track of things.

Applications of Derivatives

Three big ones
- Finding minima and maxima of a curve (the derivative is zero!)
- Finding the rate of change of some function at some point.
- Finding the equation of a ***tangent*** or a ***normal*** line to a curve at some point.

We will explore them all through one problem. Focus on the process here, not the specifics.

A baseball player hits a ball from 4 feet off the ground with an initial vertical velocity of 96 ft per sec at an angle of 45 degrees up from horizontal. The ***height*** in feet of the ball as a function of time, *t* in seconds after impact, can be modelled by the equation:

$$h(t) = -16t^2 + 92t + 4$$

When does the ball reach it's peak?

h describes height as a function of time $h(t) = -16t^2 + 96t + 4$

The derivative of *h* with respect to *t* gives the rate of change of the height of the ball. $h'(t) = -32t + 96$

At it's peak, the ball is not changing height. For what *t* is $h'(t) = 0$

t has to satisfy $0 = -32t + 96$

Solve for *t* $t = \frac{96}{32} = 3\ sec$

> Local Extremes of a function occur at values of *x* where the first derivative is zero.

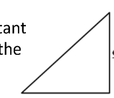
How fast is the ball moving vertically after 2 seconds?

The derivative of *h* with respect to *t* gives the rate of change (vertical speed) of the ball. $h'(t) = -32t + 96$

Just evaluate the derivative at *t* = 2 seconds $h'(2) = -32(2) + 96$
Note that the ball is still moving ***up*** (positive velocity). $= 32\ ft/sec$

> The derivative gives the rate of change of the function at any value of *x*.

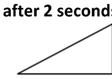
How fast is the ball moving horizontally after 2 seconds?
This ties into vectors! Assuming no air resistance, the horizontal speed is constant and is defined at the moment of impact. Using some (simple in this case) trig we find the horizontal velocity is also 96 ft/sec.



> Pay attention to the variables - *x* and *t* are different!

How fast is the ball moving after 2 seconds?

Use the vector length



Pythagorean Theorem gives $v(t) = \sqrt{96^2 + 32^2}$
$= \sqrt{10240} = 32\sqrt{10}\ ft/sec$

> Bring in other ideas as needed!

Find the equation of a line tangent to the ball's direction at 2 seconds.
To find the equation of a line we need a slope and a point!

The ratio of the vertical speed to the horizontal speed at *t* = 2 gives us the slope of the ***tangent*** line. $m = \frac{32}{96} = \frac{1}{3}$

The position of the ball at 2 sec is given by $(96t, -16t^2 + 96t + 4)$
$(96(2), -16(2)^2 + 96(2) + 4) = (192, 132)$

The tangent line can be found using point-slope form: $h - 132 = \frac{1}{3}(x - 192)$
or $h = \frac{1}{3}x + 68$

> The value of the derivative at *x* is the slope of the line tangent to the curve at *x*.

Find the equation of a line *normal* to the ball's direction at 2 seconds.
We still need a slope and a point!

The ***normal*** line is ***perpendicular*** to the ***tangent*** line. Their slopes are ***opposite reciprocals***. $m = -3$

The position of the ball at 2 sec is still $(192, 132)$

The normal line can be found using point-slope form: $h - 132 = -3(x - 192)$
or $h = -3x + 708$

> Pay attention to words - ***normal*** and ***tangent*** are different but related.

Further work with derivatives:

Chapter 18 — Applications of differential calculus
- A Time rate of change
- B General rates of change
- C Motion in a straight line
- D Some curve properties
- E Rational functions
- F Inflections and shape
- G Optimisation

Chapter 19 — Derivatives of exponential and logarithmic functions
- A Exponential e
- B Natural logarithms
- C Derivatives of logarithmic functions
- D Applications

Chapter 20 — Derivatives of trigonometric functions
- A Derivatives of trigonometric functions
- B Optimisation with trigonometry

Choose your review work as needed from these topics over the course of the next few weeks. There are 9 weeks (including Spring Break) before your exam.

I suggest the following plan for reviewing derivatives:
- Read through each section and look at the problem sets.
- Choose 4-5 problems per section - pick 2 easy, 3 hard. Do them completely.
- Do a couple more and/or get help if you find yourself struggling.
- Set a goal of covering two sections per week between now and your exam.
- Pick a fixed day of the week to do a section.
- Work with friends - but be sure that you are doing your own learning.

Key ideas by section:

18A&B Rates of change

> The derivative of a function is it's rate of change or velocity.
> Velocity has direction:
  >0 is forward,right,up; <0 is reverse,left,down; = 0 is not moving
> ***Speed*** is the absolute value of velocity or $|f'(x)|$. It is always positive.
> The second derivative is it's acceleration.
> Acceleration also has direction:
  > 0 is increasing (speeding up); < 0 is decreasing (slowing down); = 0 is constant

18C Motion along a straight line

> The position, relative to the starting point is called ***displacement***.
  >0 is right; <0 is left

18D Properties of curvature

> A curve that is always increasing or decreasing is called ***monotonic***.
> If $f'(x) > 0$ on an interval, the curve is ***monotonically increasing*** on the interval.
> If $f'(x) < 0$ on an interval, the curve is ***monotonically decreasing*** on the interval.
> Sign diagrams are a good way of characterizing a curve.
  > ***Critical values*** occur at zeros and discontinuities
    >SL only explores vertical asymptotes as discontinuities
  > The ***value*** of a function cannot change sign except at a critical value
  > The value of a function does not always change sign at a critical value

18E Rational Functions (ratios of polynomials)

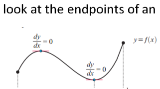
> Vertical asymptotes occur when you divide by zero (denominator = 0!)
  > Examine the sign diagram near the asymptotes to find precise behavior
> Horizontal asymptotes occur as follows:
  > If degree of the numerator > degree of the denominator
    > No horizontal asymptote. Function diverges.
  > If degree of the numerator = degree of the denominator
    > Horizontal asymptote at the ratio of the leading coefficients
    > Other situations occur - not covered in SL
  > If degree of the numerator < degree of the denominator
    > Horizontal asymptote at zero
> Zeros occur for values of x that make the numerator zero.

18F Inflections and Shape

> Second derivative measures ***curvature***.
  > $f''(x) > 0$ is concave up (smiling)
  > $f''(x) < 0$ is concave down (frowning)
  > $f''(a) = 0$ is a ***point of inflection*** at a iif the sign of f'' changes on either side of a
> A ***stationary point of inflection*** also has $f'(x) = 0$ (horizontal slope)

18G Optimization

> A local minimum or maximum must occur when $f'(a) = 0$
> No all $f'(a) = 0$ result in local min or max
  > Sign must change on either side of the value a
  > Can also look at f''
    > If $f'(a) = 0$ and $f''(a) > 0$, a is a local min
    > If $f'(a) = 0$ and $f''(a) < 0$, a is a local max
    > If $f'(a) = 0$ and $f''(a) = 0$, a is a stationary point of inflection
> Don't forget to look at the endpoints of an interval before choosing the min or max



Chapter 19 Derivatives of Exponentials and Logarithms

> Derivative of e^x is itself

> Derivative of $\ln x$ is $\frac{1}{x}$

> Don't forget the chain rule!

Chapter 20 Derivatives of Trig Functions

> Derivative of $\sin x$ is $\cos x$
> Derivative of $\cos x$ is $-\sin x$

> Derivative of $\tan x$ is $\frac{1}{\cos^2 x}$

> Don't forget the chain rule!
> Unit circle, radian measure, and values of trig function for important angles
> Various trig identities: (see Formula Sheet)
  > $\sin^2 x + \cos^2 x = 1$
  > $\sin 2x = 2\sin x\cos x$
  > $\sin(a + b) = \sin a \sin b + \cos a \cos b$
  > $\cos(a + b) = \cos a \cos b - \sin a \sin b$

Integrals

> Anti-derivatives 21A
    Reverse differentiating with attention to chain rule
    Particular case of composites with $ax + b$ 21D
> Indefinite integrals - require a constant
    Finding the constant from an initial condition or a known point
> Definite integrals
    Fundamental Theorem 21B
> Applications
    Area under and between curves 22A
    Area interpretation as probability 24A
    Area as distance 22B
    Area as volume of water added, etc. 22C
    Volumes of revolution 22D

A ANTIDIFFERENTIATION

Elliot is cruising along a straight road at 90 ft/sec when his radar detector beeps. He slams on his brakes, decelerating at a rate of 20 ft/sec^2. a) How long does it take him to get to the 20 mph speed limit (30 ft/sec)? b) How far does he travel in that time?

a) $30 = -20t + 90$ $t = 3$ seconds

b) We need a function that describes position as a function of time. Recall that velocity is the derivative of position. So first we want to find a function whose derivative is $v(t) = -20t + 90$

$$s(t) = -10t^2 + 90t + c$$

The ***antiderivative*** of a function $f(x)$ is that function whose derivative is $f(x)$.

The antiderivative of f is written as F with the property that $F'(x) = f(x)$

We find antiderivatives by
a) thinking in reverse from differentiation or
b) integration (we will look more at this later)

Some examples:

1. Solve the following *antiderivative* questions. In other words, find F, g, and S:
(a) $F'(x) = 4x^3$ **(b)** $g'(t) = 10\cos(5t)$ **(c)** $\frac{dS}{du} = \frac{1}{2}e^u + \frac{1}{2}e^{-u}$
Did you notice that there is more than one correct answer to each question?

B THE FUNDAMENTAL THEOREM OF CALCULUS

Back to our question. We found that the ***position*** of the object is given by:

$$s(t) = -10t^2 + 90t + c$$

Since Elliot has not backtracked at all, we can find the distance travelled by looking at his position at $t = 0$ and his position at $t = 3$:

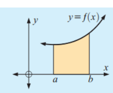
$s(0) = -10(0)^2 + 90(0) + c = c$ $s(3) = -10(3)^2 + 90(3) + c = 180 + c$

If we take the difference, we have the distance travelled:

$$s(3) - s(0) = 180 + c - c = 180 \text{ feet}$$

Let's look at this visually SLCalcSketches.gsp

If $f(x)$ is a continuous positive function on an interval $a \leqslant x \leqslant b$ then the area under the curve between $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$.



One can calculate these areas by using infinite summations of rectangles or trapezoids but it is rather complex. We will state (without proving) a very important result:

The Fundamental Theorem of Calculus (Part I)

For a continuous function $f(x)$ with antiderivative $F(x)$, $\int_a^b f(x)\,dx = F(b) - F(a)$.

In other words instead of evaluating an infinite sum using limits, we can simply evaluate the difference of the antiderivative of the function evaluated at the two endpoints of the interval.

This is a ***huge*** simplification! From this theorem, other properties can be proven:

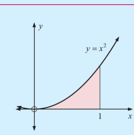
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b c\,dx = c(b-a)$ {c is a constant}
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Understanding that integrals are ***areas under a curve*** helps you remember these.

Let's try a couple:

Use the fundamental theorem of calculus to find the area:
a between the x-axis and $y = x^2$ from $x = 0$ to $x = 1$
b between the x-axis and $y = \sqrt{x}$ from $x = 1$ to $x = 9$.

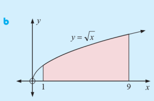
a

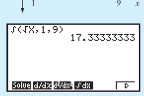


$f(x) = x^2$ has antiderivative $F(x) = \frac{x^3}{3}$

$$\therefore \text{ the area } = \int_0^1 x^2\,dx = F(1) - F(0) = \tfrac{1}{3} - 0 = \tfrac{1}{3} \text{ units}^2$$

b





$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

$$\therefore \text{ the area } = \int_1^9 x^{\frac{1}{2}}\,dx = F(9) - F(1) = \tfrac{2}{3} \times 27 - \tfrac{2}{3} \times 1 = 17\tfrac{1}{3} \text{ units}^2$$

HW	21A	#1-3
	21B	#1-4

Because we will have no class on Tuesday 3/6 (Field Trip), I have included 21C.2 in "tonight's" HW, due Thu 3/8. Basically you want to get through Ch 21 before Spring Break.

C INTEGRATION

We have been looking at finding the area under a curve over a specific interval. But the Fundamental Theorem of Calculus is more powerful than that. We can use it to relate a function and its antiderivative, even if there is no specific interval under consideration. The Fundamental Theorem implies that:

$$\text{if } F'(x) = f(x) \text{ then } \int f(x)\,dx = F(x) + c.$$

In words, the ***indefinite integral*** of a function is given by the antiderivative of the function plus some constant of integration. The function *f*(*x*) is called the ***integrand***.

Example: Consider the function $f(x) = x^2$

The antiderivative of this function is $F(x) = \frac{x^3}{3} + c$ since it's derivative gives $f(x)$

We write this as the ***integral*** of *f* or $\int x^2\,dx = \frac{x^3}{3} + c$

Notice that the same rules apply as for definite integrals, except that there is always a constant of integration. Another example.

Find $\int (2x^2 - 4x + 5)dx$

$$= \int 2x^2\,dx - \int 4x\,dx + \int 5dx$$
$$= 2\int x^2\,dx - 4\int x\,dx + \int 5dx$$
$$= \left(2\frac{x^3}{3} + c_1\right) - \left(4\frac{x^2}{2} + c_2\right) + (5x + c_3)$$
$$= \frac{2x^3}{3} - 2x^2 + 5x + c$$

Check your work! If you differentiate your result, you should get the integrand!

Try $\int \frac{18x - 6}{3x^2 - 2x + 4}dx$

Notice the reciprocal and recall that: $\frac{d(\ln x)}{dx} = \frac{1}{x}$

So we take a look at: $\frac{d\ln(3x^2 - 2x + 4)}{dx} = \frac{6x - 2}{3x^2 - 2x + 4}$

This result is a third of what we need. So if we multiply by 3, we get $F(x) = 3\ln(3x^2 - 2x + 4) + c$

Here is a summary of the integrals we know so far:

Function	*Integral*
k, a constant	$kx + c$
x^n	$\frac{x^{n+1}}{n+1} + c,\ n \neq -1$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln x + c,\ x > 0$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$

$\int k dx = kx + c$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq 1$

$\int e^x dx = e^x + c$

$\int \frac{1}{x} dx = \ln x + c$

$\int \cos x dx = \sin x + c$

$\int \sin x dx = -\cos x + c$

Using this in conjunction with what you know about differentiating, you can find many integrals. Let's try some:

Find: **a** $\int (x^3 - 2x^2 + 5)\,dx$ **b** $\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$

a $\int (x^3 - 2x^2 + 5)\,dx$
$= \frac{x^4}{4} - \frac{2x^3}{3} + 5x + c$

b $\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$
$= \int (x^{-3} - x^{\frac{1}{2}})\,dx$
$= \frac{x^{-2}}{-2} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= -\frac{1}{2x^2} - \frac{2}{3}x^{\frac{3}{2}} + c$

Always differentiate your result to see that you get the original integrand.

Integrate with respect to x:
a $2\sin x - \cos x$ **b** $-\frac{2}{x} + 3e^x$

a $\int [2\sin x - \cos x]dx$
$= 2(-\cos x) - \sin x + c$
$= -2\cos x - \sin x + c$

b $\int \left[-\frac{2}{x} + 3e^x\right] dx$
$= -2\ln x + 3e^x + c$ provided $x > 0$

Find: **a** $\int \left(3x + \frac{2}{x}\right)^2 dx$ **b** $\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx$

a $\int \left(3x + \frac{2}{x}\right)^2 dx$
$= \int \left(9x^2 + 12 + \frac{4}{x^2}\right) dx$
$= \int (9x^2 + 12 + 4x^{-2})\,dx$
$= \frac{9x^3}{3} + 12x + \frac{4x^{-1}}{-1} + c$
$= 3x^3 + 12x - \frac{4}{x} + c$

b $\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx$
$= \int \left(\frac{x^2}{\sqrt{x}} - \frac{2}{\sqrt{x}}\right) dx$
$= \int (x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})\,dx$
$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= \frac{2}{5}x^2\sqrt{x} - 4\sqrt{x} + c$

Expand parentheses and simplify to get a form you can integrate (when possible).

When you know any specific point on the curve that results from the integration, you can find the constant of integration.

Find $f(x)$ given that $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$.

Since $f'(x) = x^3 - 2x^2 + 3$,
$f(x) = \int (x^3 - 2x^2 + 3)\,dx$
$\therefore\ f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + c$
But $f(0) = 2$, so $0 - 0 + 0 + c = 2$ and so $c = 2$
Thus $f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + 2$

We may be given a second derivative, *f*''. To find the original function, integrate twice. You will need a point on *f*' to be specific on the second integration.

Find $f(x)$ given that $f''(x) = 12x^2 - 4$, $f'(0) = -1$ and $f(1) = 4$.

If $f''(x) = 12x^2 - 4$
then $f'(x) = \frac{12x^3}{3} - 4x + c$ {integrating with respect to x}
$\therefore\ f'(x) = 4x^3 - 4x + c$
But $f'(0) = -1$ so $0 - 0 + c = -1$ and so $c = -1$
Thus $f'(x) = 4x^3 - 4x - 1$
$\therefore\ f(x) = \frac{4x^4}{4} - \frac{4x^2}{2} - x + d$ {integrating again}
$\therefore\ f(x) = x^4 - 2x^2 - x + d$
But $f(1) = 4$ so $1 - 2 - 1 + d = 4$ and so $d = 6$
Thus $f(x) = x^4 - 2x^2 - x + 6$

Remember, when we see $g(x) = \int f(x)dx$ the function *f* is the ***derivative*** of *g*

HW	21C.1	#2-11
	21C.2	#1-8 last col, 9 & 10

D INTEGRATING $f(ax+b)$

This is a basic extension of integrating *f*(*x*). Notice that the derivative of *ax* + *b* is just *a*. So when we integrate a function of *ax* + *b*, we will need to divide by *a*. This is a consequence of the chain rule.

Apart from noticing this situation, there is nothing new here.

Function	*Integral*
e^{ax+b}	$\frac{1}{a}e^{ax+b}+c$
$(ax+b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}+c, \quad n\neq -1$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b)+c, \quad ax+b>0$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)+c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)+c$

E DEFINITE INTEGRALS

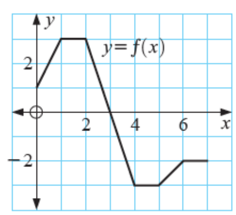
This section is practice calculating definite integrals on your calculator. The main idea is that you can find a definite integral of ***any*** function that you can enter into your calculator, even if you could never find it manually.

The section also includes some problems that remind you that interpreting an integral as area requires that you remember that ***area under the x-axis is considered negative.***

Evaluate the following integrals using area interpretation:

a $\int_0^3 f(x)\,dx$ **b** $\int_3^7 f(x)\,dx$

c $\int_2^4 f(x)\,dx$ **d** $\int_0^7 f(x)\,dx$



HW	21D	#1begh,2beh,3,4,5bcef,6cfi,7bc,8-13
	21E.1	1col2,2
	21E.2	#2-10 even & 9

Chapter 22
Applications of integration

A Finding areas between curves
B Motion problems
C Problem solving by integration
D Solids of revolution

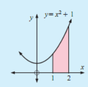
A FINDING AREAS BETWEEN CURVES

One can interpret integration as a sum of "small" areas whose height at a given *x* is *f*(*x*) and whose "width" is d*x*. Thinking of it this way helps to visualize many applications of integration.

Remember that integration honors the sign so that negative values of the function contribute negatively to the integral. If you are strictly interested in area, you must take this into consideration.

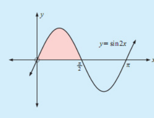
Areas between a curve and the *x*-axis are simplest. Try a couple:

Find the area of the region enclosed by $y = x^2 + 1$, the x-axis, $x = 1$ and $x = 2$.



$$\text{Area} = \int_1^2 (x^2+1)\,dx = \left[\frac{x^3}{3} + x\right]_1^2 = \left(\tfrac{8}{3}+2\right) - \left(\tfrac{1}{3}+1\right) = 3\tfrac{1}{3} \text{ units}^2$$

Find the area enclosed by one arch of the curve $y = \sin 2x$.

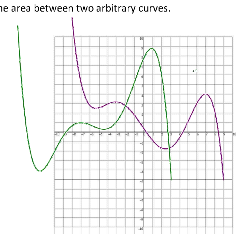


The period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$.

$\therefore$ the first positive x-intercept is $\frac{\pi}{2}$.

$$\text{The required area} = \int_0^{\frac{\pi}{2}} \sin 2x\,dx = \left[\tfrac{1}{2}(-\cos 2x)\right]_0^{\frac{\pi}{2}} = -\tfrac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{2}} = -\tfrac{1}{2}(\cos\pi - \cos 0) = 1 \text{ unit}^2$$

Now consider the area between two arbitrary curves.



Area between two curves

If the curves intersect at *x* = a and *x* = b with a > b and if *f*(*x*) ≥ *g*(*x*) on the interval [a, b], then the area between *f*(*x*) and *g*(*x*) between a and b is given by:

$$\int_a^b [f(x) - g(x)]dx$$

Find the area of the region enclosed by $y = x + 2$ and $y = x^2 + x - 2$.

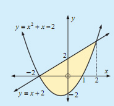
$y = x + 2$ meets $y = x^2 + x - 2$ where

$x^2 + x - 2 = x + 2$

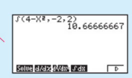
$\therefore\ x^2 - 4 = 0$

$\therefore\ (x+2)(x-2) = 0$

$\therefore\ x = \pm 2$



$$\text{Area} = \int_{-2}^{2} [y_U - y_L]\,dx = \int_{-2}^{2} [(x+2) - (x^2+x-2)]\,dx = \int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) = 10\tfrac{2}{3} \text{ units}^2$$



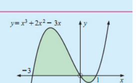
Find the total area of the regions contained by $y = f(x)$ and the x-axis for $f(x) = x^3 + 2x^2 - 3x$.

$f(x) = x^3 + 2x^2 - 3x$

$= x(x^2 + 2x - 3)$

$= x(x-1)(x+3)$

$\therefore\ y = f(x)$ cuts the x-axis at 0, 1, and −3.



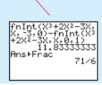
Total area

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\,dx - \int_0^1 (x^3 + 2x^2 - 3x)\,dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_0^1$$

$$= \left(0 - -11\tfrac{1}{4}\right) - \left(-\tfrac{7}{12} - 0\right)$$

$$= 11\tfrac{5}{6} \text{ units}^2$$



HW	22A.1	#1-4
	22A.2	#1-16

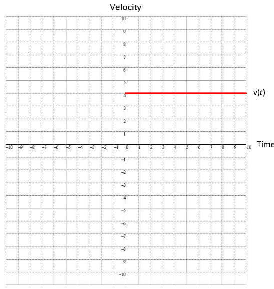
HW Guidance:

A.1 #1cd, 2dhl, 3, 4abc (do completely)
Other problems
- Sketch each graph on your calculator.
- Write each integral correctly, including limits.
- Evaluate any integrals that are unfamiliar

A.2 #1aceg, 3, 4, 10-16 all (do completely)
Other problems
- Sketch the graphs, find relevant intersections.
- Write out the integrals completely, careful with signs
- Evaluate any integrals that are unfamiliar

B MOTION PROBLEMS

The ***area*** under a velocity vs. time graph represents distance traveled.



v(*t*)*d*t* represents velocity * time = distance

But be careful, if velocity is negative, the integral will calculate ***negative*** distance traveled.

Beware of the difference between:
Distance Traveled: everything counts
Displacement: change in ***position*** from start to finish.

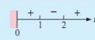
Also, recall that the derivative of velocity is acceleration. Thus, the area under an acceleration curve from *t*=a to *t*=b represents the change in velocity from *t*=a to *t*=b. More generally, the ***velocity function*** can be found by integrating the ***acceleration function***.

A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2 \text{ m s}^{-1}$.

a How far does P travel in the first 4 seconds in motion?
b Find the displacement of P after 4 seconds.

a $v(t) = s'(t) = t^2 - 3t + 2$
$= (t-1)(t-2)$

$\therefore$ sign diagram of v is:



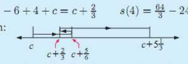
Since the signs change, P reverses direction at $t = 1$ and $t = 2$ seconds.

Now $s(t) = \int (t^2 - 3t + 2)dt = \frac{t^3}{3} - \frac{3t^2}{2} + 2t + c$

Now $s(0) = c$ $\quad s(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = c + \frac{5}{6}$

$s(2) = \frac{8}{3} - 6 + 4 + c = c + \frac{2}{3}$ $\quad s(4) = \frac{64}{3} - 24 + 8 + c = c + 5\frac{1}{3}$

Motion diagram:



$\therefore$ total distance $= \left(c + \frac{5}{6} - c\right) + \left(c + \frac{5}{6} - \left[c + \frac{2}{3}\right]\right) + \left(c + 5\frac{1}{3} - \left[c + \frac{2}{3}\right]\right)$
$= \frac{5}{6} + \frac{5}{6} - \frac{2}{3} + 5\frac{1}{3} - \frac{2}{3}$
$= 5\frac{2}{3}$ m

b Displacement $=$ final position $-$ original position
$= s(4) - s(0)$
$= c + 5\frac{1}{3} - c$
$= 5\frac{1}{3}$ m

So, the displacement is $5\frac{1}{3}$ m to the right.

C PROBLEM SOLVING BY INTEGRATION

Some other applications of integration. For a continuously changing phenomenon, the ***amount*** of change is the integral of it's ***rate*** of change over the time period in question.

A ***marginal cost*** is a cost that changes with the number of items.

The marginal cost of producing x urns per week is given by
$C'(x) = 2.15 - 0.02x + 0.000\,36x^2$ dollars per urn provided $0 \leqslant x \leqslant 120$.

The initial costs before production starts are $185. Find the total cost of producing 100 urns per day.

The marginal cost is $C'(x) = 2.15 - 0.02x + 0.000\,36x^2$ dollars per urn

$\therefore \quad C(x) = \int (2.15 - 0.02x + 0.000\,36x^2)\,dx$

$\therefore \quad C(x) = 2.15x - 0.02\frac{x^2}{2} + 0.000\,36\frac{x^3}{3} + c$
$= 2.15x - 0.01x^2 + 0.000\,12x^3 + c$

But $C(0) = 185$, so $c = 185$

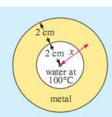
$\therefore \quad C(x) = 2.15x - 0.01x^2 + 0.000\,12x^3 + 185$

$\therefore \quad C(100) = 2.15(100) - 0.01(100)^2 + 0.000\,12(100)^3 + 185$
$= 420$ So, the total cost is $420.

A metal tube has an annulus cross-section as shown. The outer radius is 4 cm and the inner radius is 2 cm. Within the tube, water is maintained at a temperature of 100°C.

Within the metal the temperature drops off from inside to outside according to $\frac{dT}{dx} = -\frac{10}{x}$ where x is the distance from the central axis and $2 \leqslant x \leqslant 4$.

Find the temperature of the outer surface of the tube.



tube cross-section

$\frac{dT}{dx} = \frac{-10}{x}$ so $T = \int \frac{-10}{x}\,dx$

$\therefore \quad T = -10\ln x + c \qquad \{\text{since } x > 0\}$

But when $x = 2$, $T = 100$

$\therefore \quad 100 = -10\ln 2 + c$

$\therefore \quad c = 100 + 10\ln 2$

Thus $T = -10\ln x + 100 + 10\ln 2$
$T = 100 + 10\ln\left(\frac{2}{x}\right)$

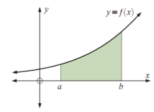
When $x = 4$, $T = 100 + 10\ln\left(\frac{1}{2}\right) \approx 93.1$

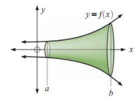
$\therefore$ the outer surface temperature is 93.1°C.

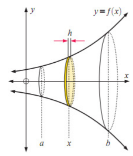
HW	22B.1	#1-3
	22B.2	#1-7
	22C	#1-4

D SOLIDS OF REVOLUTION

A common application of integration is to find volumes. In particular, the volume created when rotating a known curve around an axis is well suited to integration.





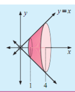


A concept to understand (not a formula to memorize!)

$$V = \lim_{h \to 0} \sum_{x=a}^{x=b} \pi [f(x)]^2 h = \int_a^b \pi [f(x)]^2 \, dx = \pi \int_a^b y^2 \, dx$$

Example 11 Self Tutor

Use integration to find the volume of the solid generated when the line $y = x$ for $1 \leqslant x \leqslant 4$ is revolved around the x-axis.



$$\begin{aligned}\text{Volume of revolution} &= \pi \int_a^b y^2 \, dx \\ &= \pi \int_1^4 x^2 \, dx \\ &= \pi \left[\frac{x^3}{3}\right]_1^4 \\ &= \pi \left(\tfrac{64}{3} - \tfrac{1}{3}\right) \\ &= 21\pi \quad \text{cubic units}\end{aligned}$$

The volume of a cone is

$V_{cone} = \frac{1}{3}\pi r^2 h$

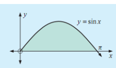
So, in this example

$$\begin{aligned}V &= \tfrac{1}{3}\pi 4^2(4) - \tfrac{1}{3}\pi 1^2(1) \\ &= \tfrac{64\pi}{3} - \tfrac{\pi}{3} \\ &= 21\pi \quad \text{which checks } \checkmark\end{aligned}$$

When possible, use geometry to check your result. Also, use symmetry, especially when there are positive and negative signs involved.

One arch of $y = \sin x$ is rotated about the x-axis.

What is the volume of revolution?



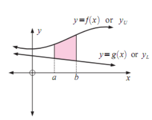
$$\begin{aligned}\text{Volume} &= \pi \int_a^b y^2 \, dx \\ &= \pi \int_0^\pi \sin^2 x \, dx \\ &= \pi \int_0^\pi \left[\tfrac{1}{2} - \tfrac{1}{2}\cos(2x)\right] dx \\ &= \pi \left[\tfrac{x}{2} - \tfrac{1}{2}\left(\tfrac{1}{2}\right)\sin(2x)\right]_0^\pi \\ &= \pi \left[\left(\tfrac{\pi}{2} - \tfrac{1}{4}\sin(2\pi)\right) - \left(0 - \tfrac{1}{4}\sin 0\right)\right] \\ &= \pi \times \tfrac{\pi}{2} \\ &= \tfrac{\pi^2}{2} \text{ units}^3\end{aligned}$$

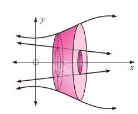
Remember

$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

With two functions, one can calculate more complex volumes:





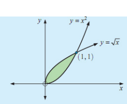
Extending the concept, the volume generated by revolving the area between two curves around the *y*-axis is given by:

$$\int_a^b \pi [f_1(x)]^2 \, dx - \int_a^b \pi [f_2(x)]^2 \, dx$$

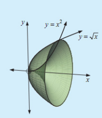
$$= \pi \int_a^b \left\{[f_1(x)]^2 - [f_2(x)]^2\right\} dx$$

Do not memorize this! Understand what it means and set up each situation as needed.

Find the volume of revolution generated by revolving the region between $y = x^2$ and $y = \sqrt{x}$ about the x-axis.



$$\begin{aligned}\text{Volume} &= \pi \int_0^1 \left(y_U{}^2 - y_L{}^2\right) dx \\ &= \pi \int_0^1 \left((\sqrt{x})^2 - (x^2)^2\right) dx \\ &= \pi \int_0^1 (x - x^4) \, dx \\ &= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 \\ &= \pi \left(\left(\tfrac{1}{2} - \tfrac{1}{5}\right) - (0)\right) \\ &= \tfrac{3\pi}{10} \text{ units}^3\end{aligned}$$



HW	22D.1	#1bdfh,2,3b,4,5,6,8,9
	22D.2	#1-6