Mon 5/7	7A 1gdh,2dh,3cd,5bc,6,7,8ab	Binomial Expansions
Wed 5/9	7B 1c,2cd,3bd,4,6,7,8,9	Binomial Theorem
Fri 5/11	Practice: QB Problems #1-10 (Test Monday 5/14)	

Chapter 7
The binomial expansion

A BINOMIAL EXPANSIONS

The sum $a + b$ is called a **binomial** as it contains two terms.
Any expression of the form $(a + b)^n$ is called a **power of a binomial**.

We have looked at the simplest binomial expansion many times:

$(a + b)^2 = \quad a^2 + 2ab + b^2$

Let's do a few more and look for a pattern:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) \\ &= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 \\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

$a^2 + 2ab + b^2$ is the **binomial expansion** of $(a + b)^2$
$a^3 + 3a^2b + 3ab^2 + b^3$ is the **binomial expansion** of $(a + b)^3$

Notice anything yet? Let's do one more

$$\begin{aligned}(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4 \\ &= a^4+4a^3b^1+6a^2b^2+4a^1b^3+b^4\end{aligned}$$

Can you guess what $(a + b)^5$ would be?
(*Hint*: Start with the powers of *a* and *b*, then look for a pattern in the coefficients.)

Pascal's Triangle

n = 0 1
n = 1 1 1
n = 2 1 2 1
n = 3 1 3 3 1
n = 4 1 4 6 4 1
n = 5 1 5 10 10 5 1
n = 6 1 6 15 20 15 6 1
r = 0 1 2 3 4 5 6

To get the next row, add the two numbers diagonally above

In general for the expansion of $(a + b)^n$ where *n* is a positive integer:
> It is a sum of terms which are the product of a ***coefficient***, a ***power of a***, and a ***power of b***.
> The coefficients come from Pascal's Triangle. The coefficient of $a^k b^{n-k}$ is ${}_nC_k$
> The powers of *a* begin at *n* and ***decrease*** by one for each successive term
> The powers of *b* begin at 0 and ***increase*** by one for each successive term
> The sum of the exponents in the powers of *a* and *b* for each term is *n*
> There are *n* + 1 terms

Try $(x + 4)^6$ $= x^6 + 6x^5(4) + 15x^4(4^2) + 20x^3(4^3) + 15x^2(4^4) + 6x(4^5) + 4^6$
$= x^6 + 24x^5 + 960x^4 + 1280x^3 + 3840x^2 + 6144x + 4096$

In most cases the values of "*a*" and "*b*" are a little more complex:

Try $(x - 5)^4$ $= x^4 + 4x^3(-5) + 6x^2(-5)^2 + 4x(-5)^3 + (-5)^4$
$= x^4 - 20x^3 + 150x^2 - 500x + 625$

$(2x + 4y)^3$ $= x^3 + 3(2x)^2(4y) + 3(2x)(4y)^2 + (4y)^3$
$= x^3 + 48x^2y + 96xy^2 + 64y^3$

Some advice *(ie. why this class is not independent study...)*:
> Write out the full expansion using the coefficients, and parentheses around each power.
> Systematically and carefully go through each term to expand out the powers, multiplying the results.
> Take pains to identify when you are ***adding***, ***subtracting***, ***multiplying*** or using ***exponents***

Find the binomial expansion of $\left(x - \frac{2}{x}\right)^5$.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Letting $a = (x)$ and $b = \left(\frac{-2}{x}\right)$, we find

$$\begin{aligned}\left(x - \tfrac{2}{x}\right)^5 &= (x)^5 + 5(x)^4\left(\tfrac{-2}{x}\right) + 10(x)^3\left(\tfrac{-2}{x}\right)^2 + 10(x)^2\left(\tfrac{-2}{x}\right)^3 \\ &\quad + 5(x)\left(\tfrac{-2}{x}\right)^4 + \left(\tfrac{-2}{x}\right)^5 \\ &= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}\end{aligned}$$

HW Hints:
#6b: Do this without a calculator. Use your calculator to check your answer.
#8: Try to find just the term you want.

7A 1gdh,2dh,3cd,5bc,6,7,8ab

B THE BINOMIAL THEOREM

Recall that the coefficients in a binomial expansion come from Pascal's Triangle (Remember Lacsap?)

These numbers can be calculated using a formula that will reappear often. It describes the number of ways that you can select *r* items from a set of *n* distinct items without considering their order and is called "***n* choose *r***"

The element in row *n* column *r* of Pascal's Triangle is given by: $C_r^n = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Find ${}_6C_2$ using the ! button (MATH/PRB/! (Option 4):

Pascal's Triangle

$n = 0$: 1
$n = 1$: 1 1
$n = 2$: 1 2 1
$n = 3$: 1 3 3 1
$n = 4$: 1 4 6 4 1
$n = 5$: 1 5 10 10 5 1
$n = 6$: 1 6 15 20 15 6 1
$r =$ 0 1 2 3 4 5 6

A manual way to find ${}_nC_r$ that's easy and worth knowing: ${}_nC_r = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}$

${}_6C_2 = \frac{6\cdot 5}{1\cdot 2}$ Try ${}_{12}C_5 = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 11\cdot 72 = 792$

Even better! Use the the ${}_nC_r$ button (MATH/PRB/nCr (Option 3): Enter *n*, then nCr, then *r*.

The Binomial Theorem

defines the expansion of a binomial

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{r}a^{n-r}b^r + ... + b^n$$

where $\binom{n}{r}$ is the **binomial coefficient** of $a^{n-r}b^r$ and $r = 0, 1, 2, 3, ..., n$.

or, more concisely: $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r}a^{n-r}b^r$

Notice that the first term uses $r = 0$ so

The **general term** or $(r+1)$th term in the binomial expansion is $T_{r+1} = \binom{n}{r}a^{n-r}b^r$.

Some applications:

Write down the first 3 and last 2 terms of the expansion of $\left(2x+\frac{1}{x}\right)^{12}$.

$$\left(2x+\frac{1}{x}\right)^{12} = (2x)^{12} + \binom{12}{1}(2x)^{11}\left(\frac{1}{x}\right) + \binom{12}{2}(2x)^{10}\left(\frac{1}{x}\right)^2 + ...$$
$$...... + \binom{12}{11}(2x)\left(\frac{1}{x}\right)^{11} + \left(\frac{1}{x}\right)^{12}$$

Find the 7th term of $\left(3x - \frac{4}{x^2}\right)^{14}$. Do not simplify your answer.

$a = (3x)$, $b = \left(\frac{-4}{x^2}\right)$ and $n = 14$

So, as $T_{r+1} = \binom{n}{r}a^{n-r}b^r$, we let $r = 6$

$\therefore \quad T_7 = \binom{14}{6}(3x)^8\left(\frac{-4}{x^2}\right)^6$

In the expansion of $\left(x^2+\frac{4}{x}\right)^{12}$, find:

a the coefficient of x^6

b the constant term

$a = (x^2)$, $b = \left(\frac{4}{x}\right)$ and $n = 12$

$\therefore \quad T_{r+1} = \binom{12}{r}(x^2)^{12-r}\left(\frac{4}{x}\right)^r$
$= \binom{12}{r}x^{24-2r}\frac{4^r}{x^r}$
$= \binom{12}{r}4^r x^{24-3r}$

a If $24 - 3r = 6$
then $3r = 18$
$\therefore \quad r = 6$
$\therefore \quad T_7 = \binom{12}{6}4^6x^6$
$\therefore$ the coefficient of x^6 is
$\binom{12}{6}4^6$ or 3 784 704.

b If $24 - 3r = 0$
then $3r = 24$
$\therefore \quad r = 8$
$\therefore \quad T_9 = \binom{12}{8}4^8x^0$
$\therefore$ the constant term is
$\binom{12}{8}4^8$ or 32 440 320.

Find the coefficient of x^5 in the expansion of $(x+3)(2x-1)^6$.

$(x+3)(2x-1)^6$
$= (x+3)\left[(2x)^6 + \binom{6}{1}(2x)^5(-1) + \binom{6}{2}(2x)^4(-1)^2 + ...\right]$
$= (x+3)\left(2^6x^6 - \binom{6}{1}2^5x^5 + \binom{6}{2}2^4x^4 - ....\right)$

(1) (2)

So, the terms containing x^5 are $\binom{6}{2}2^4x^5$ from (1)
and $-3\binom{6}{1}2^5x^5$ from (2)

$\therefore$ the coefficient of x^5 is $\binom{6}{2}2^4 - 3\binom{6}{1}2^5 = -336$

7B 1c,2cd,3bd,4,6,7,8,9