Chapter 6

Quadratic equations and functions

A QUADRATIC EQUATIONS

Quadratic equations are equations where the highest power of the variable (usually *x*) is two!

Applications:
> Falling bodies
> Accelleration
> Optimization

Quadratic equations come in different flavors (forms)

Standard form:	$ax^2 + bx + c = 0$	$-16t^2 - 16t + 96 = 0$
Factored form:	$(a\,x - r_1)(bx - r_2) = 0$	$(4x + 5)(2x - 3) = 0$
Vertex Form:	$a(x - h)^2 + k = 0$	$-3(x + 4)^2 + 6 = 0$

Solving quadratic equations (like any equation) means...finding the value(s) of the variable that make it true! These values are called ***solutions*** (you will also see the terms ***roots*** or ***zeros***.)

Doing that for quadratics is not as easy as for linear equations - we cannot just manipulate the sides!

There are several techniques that are all helpful in different situations:
> Factoring
> Graphing
> Completing the square
> Using a formula

We will look at these in 6A, beginning with factoring.

Look at a simple equation in factored form: (*x* - 1)(*x* - 2) = 0 (Remember what a ***factor*** is?!)

We know that if two ***factors*** multiply to zero, one of them must equal zero. This is called the ***zero product property.***

So if (*x* - 1)(*x* - 2) = 0, then either (*x* - 1) = 0 or (*x* - 2) = 0. These are two ***linear*** equations which we can easily solve using our linear manipulations. Clearly, *x* = 1 or *x* = 2 are the two values we need. So notice that:

A quadratic equation can have two solutions!

If quadratic functions were always given in factored form, there would not be much to discuss. But consider a function in standard form.

This is not so friendly. We can try to ***factor*** the left hand side to find the solutions.

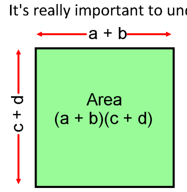
$$x^2 - 6x + 5 = 0$$

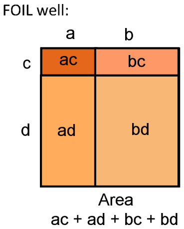
In this case, that's not too hard to do. We have to remember ***FOIL*** and do it in ***reverse***.

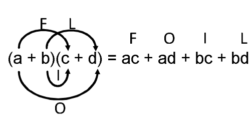
$$(x - 1)(x - 5) = 0$$

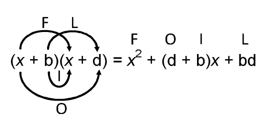
So, *x* is either 5 or 1.

It's really important to understand FOIL well:









Try a few:

$a^2 - 13a + 22$	$q^2 - 11q + 28$	$x^2 - 7x - 18$	$m^2 + 8m - 65$
(a − 11)(a − 2)	**(q − 7)(q − 4)**	**(x − 9)(x + 2)**	**(m + 13)(m − 5)**

Wouldn't it be lovely if all equations were that easy to factor? Well, yes, but they wouldn't be very useful.
So look at some other situations: (*a*, *b*, and *c* represent the coefficients from standard form)

When *b or c* = 0

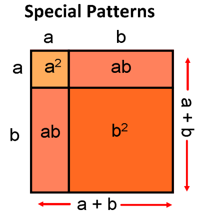
$12x^2 - 15x = 0 \qquad 3x(4x - 5) = 0 \qquad \text{so } x = 0 \text{ or } \frac{5}{4}$

one of the solutions will be zero!

$a^2 - 49 = 0$ **−7, 7** $\qquad m^2 = 7m$ **0, 7** $\qquad u^2 = -9u$ **0, −9** $\qquad n^2 - 6n = 0$ **0, 6**

When *a* is not equal to one: Several options - takes some experience
> Look for a special pattern
> Check to see if it's factorable
> Directed guess and check (or use the British Method)

Special Patterns

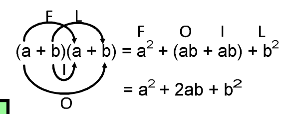


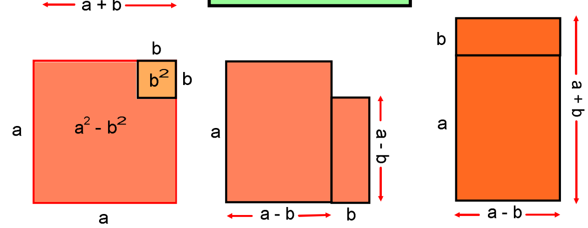
A = $(a + b)^2$

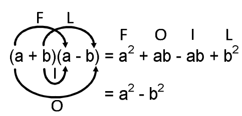
A = $a^2 + 2ab + b^2$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$







$a^2 - b^2 = (a + b)(a - b)$

Pattern	Example
$a^2 - b^2 = (a + b)(a - b)$	$x^2 - 4 = (x + 2)(x - 2)$
$a^2 + 2ab + b^2 = (a + b)^2$	$x^2 + 6x + 9 = (x + 3)^2$
$a^2 - 2ab + b^2 = (a - b)^2$	$x^2 - 4x + 4 = (x - 2)^2$

You can always do these with reverse FOIL (and you should check). But you will save a ***ton*** of time if you learn these.

$b^2 - 81$ $\mathbf{(b - 9)(b + 9)}$ $\qquad x^2 - 24x + 144$ $\mathbf{(x - 12)^2}$ $\qquad c^2 + 28c + 196$ $\mathbf{(c + 14)^2}$

Special patterns are more useful when the ***leading coefficient*** *(a)* is not one.

$9p^2 - 12p + 4$ $\mathbf{(3p - 2)^2}$ $\qquad 64w^2 + 144w + 81$ $\mathbf{(8w + 9)^2}$ $\qquad 49n^2 - 16$ $\mathbf{(7n - 4)(7n + 4)}$

Remember that ***factoring is a step*** in solving quadratic equations. Sometimes you have to manipulate first.

$4x^2 + 1 = 4x \qquad x = ½$

1. $(x+4)^2$	$x^2+8x+16$
2. $(x+9)^2$	$x^2+18x+81$
3. $(x+\frac{1}{2})^2$	$x^2+x+\frac{1}{4}$
4. $(2x+1)^2$	$4x^2+4x+1$
5. $(3x+3)^2$	$9x^2+18x+9$
6. $(\frac{3}{4}x+\frac{1}{3})^2$	$\frac{9}{16}x^2+\frac{1}{2}x+\frac{1}{9}$
7. $(x-5)^2$	$x^2-10x+25$
8. $(x-7)^2$	$x^2-14x+49$
9. $(x-\frac{1}{3})^2$	$x^2-\frac{2}{3}x+\frac{1}{9}$
10. $(4x-2)^2$	$16x^2-16x+4$
11. $(2x-6)^2$	$4x^2-24x+36$
12. $(\frac{1}{3}x-\frac{3}{4})^2$	$\frac{1}{9}x^2-\frac{1}{2}x+\frac{9}{16}$
13. $(x+2)(x-2)$	x^2-4
14. $(x+4)(x-4)$	x^2-16
15. $(x+\frac{1}{2})(x-\frac{1}{2})$	$x^2-\frac{1}{4}$
16. $(2x+3)(2x-3)$	$4x^2-9$
17. $(6x-5)(6x+5)$	$36x^2-25$
18. $(\frac{3}{4}x+\frac{1}{3})(\frac{3}{4}x-\frac{1}{3})$	$\frac{9}{16}x^2-\frac{1}{9}$
19. $(4x-9)(4x+9)$	$16x^2-81$
20. $(x-11)^2$	$x^2-22x+121$
21. $(2x+12)^2$	$4x^2+48x+144$
22. $(4x-15)^2$	$16x^2-120x+225$
23. $(8x+3)^2$	$64x^2+48x+9$
24. $(\frac{3}{4}x+\frac{1}{3})(\frac{3}{4}x-\frac{1}{3})$	$\frac{9}{16}x^2-\frac{1}{9}$

Sometimes, you'll have a "hidden" special pattern

Always Factor out the Greatest Common Factor (GCF) first **!**

Example: Factor $3x^2 + 12x - 15$

Notice that there is a common factor in all three terms!

When there is you should always factor it out first

$3x^2 + 12x - 15$

$= 3(x^2 + 4x - 5)$

$= 3(x + 5)(x - 1)$

Or consider

$5x^3 - 15x^2 - 90x$

$= 5x(x^2 - 3x - 18)$

$= 5x(x - 6)(x + 3)$

What if GCF and Special Patterns don't work?

Example: Solve $3x^2 + 10x - 8 = 0$

Can I ***factor*** $3x^2 + 10x - 8$? Well, hmmm...

One way is to guess and check! (it can be tedious, but not always)

Can you reverse think FOIL? (___x + ___)(___x + ___)

The FIRSTS have to multiply to give 3. So they must be 1 & 3

The LASTS have to have different signs and multiply to give 8. Possibilities?
-1 & 8, 1 & -8, -2 & 4, 2 & -4

The OUTERS plus INNERS create the x term.

So the only possibilities are:
$(3x - 1)(x + 8)$ or $(3x + 8)(x - 1)$
$(3x + 1)(x - 8)$ or $(3x - 8)(x + 1)$
$(3x - 2)(x + 4)$ or $(3x + 4)(x - 2)$
$(3x + 2)(x - 4)$ or $(3x - 4)(x + 2)$

Which one will add to a middle term of $+10x$? That's your factorization.

In this case, it's $(3x - 2)(x + 4)$ so the solutions to the equation are 2/3 and -4.

That's all well and good. But what about:
$15x^2 - 2x - 8 = 0$

The firsts have to multiply to give 15. So they must be 15 & 1 or 5 & 3

The lasts have to have different signs and multiply to give 8

-1 & 8, 1 & -8, -2 & 4, 2 & -4

So the "only" possibilities are:

$(15x - 1)(x + 8)$ or $(15x + 8)(x - 1)$
$(15x + 1)(x - 8)$ or $(15x - 8)(x + 1)$
$(15x - 2)(x + 4)$ or $(15x + 4)(x - 2)$
$(15x + 2)(x - 4)$ or $(15x - 4)(x + 2)$

$(5x - 1)(3x + 8)$ or $(5x + 8)(3x - 1)$
$(5x + 1)(3x - 8)$ or $(5x - 8)(3x + 1)$
$(5x - 2)(3x + 4)$ or $(5x + 4)(3x - 2)$
$(5x + 2)(3x - 4)$ or $(5x - 4)(3x + 2)$

Isn't this going a bit far?

When a and c are not prime numbers, there is a better way!

Reconsider $15x^2 - 2x - 8 = 0$

1) Calculate *ac* $ac = -120$
2) Find a magic pair that multiplies to ***ac*** and adds to *b* -12 & 10
3) Rewrite the original expression using a sum for *b*
$= 15x^2 - 12x + 10x - 8$
4) Factor by grouping: (do the "*Groupie Groupie*")
Factor out the ***GCF*** from the first two terms $= 3x(5x - 4) + 10x - 8$
Factor the "twin" from the last two terms! $= 3x(5x - 4) + 2(5x - 4)$
Gather the *GOOP* (**G**arbage **O**utside **O**f **P**arentheses) $= (3x + 2)(5x - 4)$
5) Check using FOIL
$(3x + 2)(5x - 4) = 15x^2 + 10x - 12x - 8 = 15x^2 - 2x - 8$ Voila!

This is also known as the *British Method*.

Let's try that again....
Example 2: Solve $12x^2 + 5x - 7 = 0$

1) Calculate *ac* $ac = -84$

2) Find a magic pair that multiplies to ***ac*** and adds to *b* 12 & -7

3) Rewrite the original expression using a sum for *b*
$= 12x^2 + 12x - 7x - 7$

4) Factor by grouping: (do the "*Groupie Groupie*")
Factor out the ***GCF*** from the first two terms $= 12x(x + 1) - 7x - 7$

Factor the "twin" from the last two terms! $= 12x(x + 1) - 7(x + 1)$

Gather the *GOOP* (**G**arbage **O**utside **O**f **P**arentheses) $= (12x - 7)(x + 1)$

5) Check using FOIL
$(12x - 7)(x + 1) = 12x^2 - 7x + 12x - 7 = 12x^2 + 5x - 7$ Yay!

6) Find the ***solutions to the equation***:
$(12x - 7)(x + 1) = 0$ when $x = -1$ or when $12x - 7 = 0$......$x = 7/12$

Try one:

Solve $8x^2 - 6x - 9 = 0$

Summary of Factoring Process

1.) Check to see if you can factor (discriminant)
2) Factor out a GCF
3) Look for special patterns
4) Use reverse FOIL if a = 1
5) Use Guess & Check or British Method if a $\neq$ 1

Factoring $ax^2 + bx + c$ when $a \neq 1$

	Example	Example	Example	Example	Example
Step	$6x^2 + 17x + 12$	$-10x^2 + 14x - 4$	$6x^2 + 18x + 12$	$4x^2 + 20x + 25$	$28x^2 - 63$
1. Find the discriminant first. If it's not a perfect square, stop! You can't factor into integers!	$b^2 - 4ac = 289 - 288$ $= 1$… Keep going!	$b^2 - 4ac = 196 - 160$ $= 36$ …Keep going!	$b^2 - 4ac = 324 - 288$ $= 36$ Keep going!	$b^2 - 4ac = 400 - 400$ $= 0$ … Keep going!	$b^2 - 4ac = 0 + 7056$ $= 7056 = (84^2)$… Keep going!
2. Factor out a –1 and the Greatest Common Factor from all terms.	There is no GCF	$= -2\,(5x^2 - 7x + 2)$ *(Honey, I shrunk the kids!)*	$= 6\,(x^2 + 3x + 2)$	There is no GCF	$= 7(4x^2 - 9)$
3. Is this a special pattern?	No	No	No	Yes!	Yes!
4. Compute **ac** and **b**	**ac** = 72, **b** = 17	**ac** = 10, **b** = –7	Hey, **a** = 1!	$= (2x + 5)^2$	$= 7(2x + 3)(2x - 3)$
5. Find a magic pair that multiplies to give **ac** and adds to give **b**. If no pair exists, the expression cannot be factored using integers. Be sure to check them **all** before reaching this conclusion.	**ac = 72** | **b = 17** 1, 72 | 2, 36 | 3, 24 | 4, 18 | 6, 12 | 8, 9 | *	**ac = 10** | **b = – 7** 1, 10 | 2, 5 | –2, –5 | * List *all* the factors of **ac** and find two that add to give **b**.	So **ac** is just **c** and we use the simple method. What multiplies to give **c** and adds to give **b?** (*Answer: 2 and 1*) $= 6\,(x + 2)(x + 1)$	**Finished!** (aren't you glad you practiced those special patterns!)	**Finished!** (aren't you glad you practiced those special patterns!)
6. Rewrite the original expression with four terms using the magic pair instead of **b**.	$= 6x^2 + 9x + 8x + 12$ Don't worry about the order of the two x terms. It doesn't matter.	$= -2\,[5x^2 - 2x - 5x + 2]$ If you multiply this out at this step, you should get the original expression. Do you?	**Finished!**	*Mathematically speaking, steps 6 & 7 combined are called "factor by grouping".*	
7. Factor out the GCF from the first two terms.	$= \mathbf{3x(2x + 3)} + 8x + 12$ *(Let me introduce my twin!)*	$= -2\,[\mathbf{x(5x - 2)} - 5x + 2]$	*Do the Groupie Groupie!*		
8. Factor out the GCF from the last two terms. Factor out a –1 if the third term is negative.	$= 3x(2x + 3) + \mathbf{4(2x + 3)}$ $= 3x\,(2x + 3) + 4\,(2x + 3)$	$= -2\,[x(5x - 2) - \mathbf{1(5x - 2)}]$ Since the 3rd term ($-5x$) was negative, we factor out a –1.			
9. You end up with the same binomial. Factor out the binomial.	$= (3x + 4)(2x + 3)$	$= -2\,(x - 1)\,\mathbf{(5x - 2)}$ *Don't forget the –2 sitting in the peanut gallery …*	*Gather GOOP – Garbage Outside Of Parentheses and the twin!*		
10. Check your work by multiplying with FOIL.	$= 6x^2 + 17x + 12$	$= -2(5x^2 - 7x + 2)$ $= -10x^2 + 14x - 4$			

When done, re-read the problem. Are you *solving an equation*? If so, there is another step or two. If you're just *factoring*, you're done.

Factoring Practice: a<>1

1. $2x^2-5x-12$	6. $3x^2+10x+8$	11. $12x^2+7x+1$	16. $3x^2-8x-16$
2. $9x^2-15x+4$	7. $3x^2-7x+4$	12. $3x^2+7x+4$	17. $3x^2-11x-4$
3. $2x^2-7x+3$	8. $8x^2-18x+9$	13. $9x^2-15x+4$	18. $2x^2+5x-3$
4. $2x^2-x-6$	9. $12x^2+25x+12$	14. $3x^2+13x+12$	19. $2x^2+3x-2$
5. $3x^2+13x+4$	10. $9x^2-24x+16$	15. $2x^2+9x+4$	20. $6x^2+11x+3$

1.	$(2x+3)(x-4)$
2.	$(3x-1)(3x-4)$
3.	$(2x-1)(x-3)$
4.	$(x-2)(2x+3)$
5.	$(3x+1)(x+4)$
6.	$(x+2)(3x+4)$
7.	$(x-1)(3x-4)$
8.	$(4x-3)(2x-3)$
9.	$(3x+4)(4x+3)$
10.	$(3x-4)(3x-4)$
11.	$(4x+1)(3x+1)$
12.	$(3x+4)(x+1)$
13.	$(3x-4)(3x-1)$
14.	$(x+3)(3x+4)$
15.	$(2x+1)(x+4)$
16.	$(3x+4)(x-4)$
17.	$(3x+1)(x-4)$
18.	$(2x-1)(x+3)$
19.	$(x+2)(2x-1)$
20.	$(2x+3)(3x+1)$

Solve the equation

32. $16x^2-1=0$ $-\frac{1}{4}, \frac{1}{4}$ **33.** $11q^2-44=0$ $-2, 2$ **34.** $14s^2-21s=0$ $0, 1\frac{1}{2}$

35. $45n^2+10n=0$ $0, -\frac{2}{9}$ **36.** $4x^2-20x+25=0$ $2\frac{1}{2}$ **37.** $4p^2+12p+9=0$ $-1\frac{1}{2}$

38. $15x^2+7x-2=0$ $\frac{1}{5}, -\frac{2}{3}$ **39.** $6r^2-7r-5=0$ $-\frac{1}{2}, 1\frac{2}{3}$ **40.** $36z^2+96z+15=0$ $-2\frac{1}{2}, -\frac{1}{6}$

Sometimes the equation doesn't come so nicely packaged. You need to recognize ***when*** an equation is quadratic and be able to rewrite it in a familiar form. Are these equations quadratic? If so, write them in standard form.

$x^2 + (3k-1)x + (2k+10) = 0$

$4x^2 - 3x + x(5-4x) = 6 + \frac{2}{x}$

$x(x^2 - 5) = 3x + 5x^2 + x^3$

$\frac{2x^2(x+2)}{x} = 5x^2 - \frac{3}{x}$

$\frac{x+2}{x-9} = 4x$

$3 + \frac{5}{x} = 2x$

We could go on, but you get the point! Accurate algebraic manipulations are ***key***!

HW	6A.1	1aegikl,2behk,3bdf
	QB	1, 9

So you can factor an equation like $x^2 - 5x - 84 = 0$ into and find solutions

Suppose now, that the equation is $x^2 - 81 = 0$. Would you factor?

How about $x^2 - 12 = 0$. What are the solutions now?

Now look at $m^2 - k = 0$. Solutions?

What about $(x - 3)^2 - 4 = 0$?

Graph the function $f(x) = (x - 3)^2 - 4$

Or even $3(t + 5)^2 + 4 = 13$

What is common about all of these situations?

Solving by Square Roots

When the equation is written with no linear term, you can solve by taking a square root.

So let's try something like $x^2 + 6x = 16$. Can we take a square root of both sides?

What would you do so that you ***could*** take the square root of both sides?

By adding 9 to both sides we get a nice result: $x^2 + 6x + 9 = 25$ or $(x + 3)^2 = 25$

Now we can take the square root to get: $x = \pm 5 - 3$ or $x = 2$ or -8

(a) $x^2 - 12x + 36$ (b) $x^2 + 14x + 49$ (c) $x^2 - 20x + 100$

As suggested, these should all look like either $(x - r)^2$ or $(x + r)^2$. State the important connection between the *coefficients* of the given trinomials and the values you found for r.

5. (Continuation) In the following, choose k to create a perfect-square trinomial:

(a) $x^2 - 16x + k$ (b) $x^2 + 10x + k$ (c) $x^2 - 5x + k$

$x^2 + bx + k$ $x^2 - bx + k$

Solve these equations by ***completing the square***.

(a) $x^2 - 8x = 3$ (b) $x^2 + 10x = 11$ (c) $x^2 - 5x - 2 = 0$ (d) $x^2 + 1.2x = 0.28$

Completing the square

Rewrite an equation so that there is no linear term! Get the variable into a "perfect square".

$\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$ may come in handy!

6.A.3 The Quadratic Formula

Let's use completing the square to find a formula for the solution to ***any*** quadratic equation!

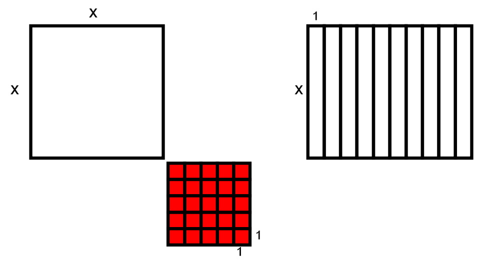
You need to be able to recreate this: Take careful notes and follow carefully:

Start with standard form:	$ax^2+bx+c=0$
Divide both sides by a:	$x^2+\frac{b}{a}x+\frac{c}{a}=0$
Subtract $\frac{c}{a}$ from both sides:	$x^2+\frac{b}{a}x=-\frac{c}{a}$
Complete the square on the LHS:	$x^2+\frac{b}{a}x+\left(\frac{1}{2}\frac{b}{a}\right)^2=-\frac{c}{a}+\left(\frac{1}{2}\frac{b}{a}\right)^2$
Clean up the RHS:	$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\left(\frac{b}{2a}\right)^2-\frac{c}{a}$ $=\frac{b^2}{4a^2}-\frac{c}{a}\cdot\frac{4a}{4a}$ $=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$ $=\frac{b^2-4ac}{4a^2}$
Now since the LHS is a perfect square:	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Time to take the square root:	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Get x all alone:	$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Remove the denominator in the radical:	$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$ $=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$ $=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
The Quadratic Formula:	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

For any quadratic equation in standard form $ax^2+bx+c=0$ the solutions, x, are given by

The Quadratic Formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

HW6A.2	1cfi,2cfi,3bcef	Complete the square!
6A.3	1beh,2bcef	(by hand!)
QB	11, 15	


x
x
1
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1

B THE DISCRIMINANT OF A QUADRATIC

Recall the quadratic formula. Let's look at it in more detail.

For any quadratic equation in standard form $ax^2+bx+c=0$ the solutions, x, are given by

The Quadratic Formula: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Let's do an example:

$3x^2-4x+5=0$

Solution: $x=\dfrac{+4\pm\sqrt{(-4)^2-4(3)(5)}}{2(3)}=\dfrac{4\pm\sqrt{16-60}}{6}$

$=\dfrac{4\pm\sqrt{-44}}{6}$

$=\dfrac{4\pm 2i\sqrt{11}}{6}$

$=\dfrac{2\pm i\sqrt{11}}{3}$

$or\ \dfrac{2+i\sqrt{11}}{3}\ and\ \dfrac{2-i\sqrt{11}}{3}$

Which leads us to looking at the expression inside the radical: Notice that:

If $b^2 - 4ac > 0$, the radical is real. There are two real solutions.
If $b^2 - 4ac < 0$, the radical is imaginary. There are two complex solutions.

If $b^2 - 4ac = 0$, the radical is zero. There is one real solution $x=\dfrac{-b}{2a}$.

If $b^2 - 4ac$ is a perfect square, you can factor the quadratic into integers!
If not, you can't!

$b^2 - 4ac$ is called the ***discriminant*** because it discriminates between types of solutions.
It's nice to know what's going to happen before you get started!

IB uses the symbol Δ to represent the discriminant

Try some:

Find the discriminant of the quadratic equation and give the number and type of solutions of the equation.

4. $2x^2+4x-4=0$ **5.** $3x^2+12x+12=0$ **6.** $8x^2=9x-11$

7. $7x^2-2x=5$ **8.** $4x^2+3x+12=3-3x$ **9.** $3x-5x^2+1=6-7x$

Another possible situation:

For the equation $kx^2+(k+3)x=1$ find the discriminant Δ and draw a sign diagram for it. Hence, find the value of k for which the equation has:

a two distinct real roots **b** two real roots

c a repeated root **d** no real roots.

For $kx^2+(k+3)x-1=0$, $a=k$, $b=(k+3)$, $c=-1$

So, $\Delta=b^2-4ac$
$=(k+3)^2-4(k)(-1)$
$=k^2+6k+9+4k$
$=k^2+10k+9$
$=(k+9)(k+1)$

and has sign diagram:

a For two distinct real roots, $\Delta>0$ $\therefore$ $k<-9$ or $k>-1$.

b For two real roots, $\Delta\geqslant 0$ $\therefore$ $k\leqslant -9$ or $k\geqslant -1$.

c For a repeated root, $\Delta=0$ $\therefore$ $k=-9$ or $k=-1$.

d For no real roots, $\Delta<0$ $\therefore$ $-9<k<-1$.

A quadratic equation $9x^2 - 5x + k = 0$ has exactly one solution. Find k.

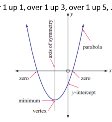
HW 6B #2 all, 3 all, 4bdf
QB 12, 14

C GRAPHING QUADRATIC FUNCTIONS

When we graph a quadratic ***function*** we get a...***parabola***.

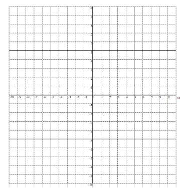
Features of the Mother function $y = x^2$
> Vertex at origin
> Pattern from origin is over 1 up 1, over 1 up 3, over 1 up 5, ... over 1 up by odds

Features of ***all*** parabolas
> Symmetric



Graphing from different forms:
Factored Form $y = a(x - p)(x - q)$
Vertex Form $y = a(x - h)^2 + k$
Standard form $y = ax^2 + bx + c$

Graph $f(x) = 2(x - 4)(x + 2)$



Zeros are easy to see!

Find the y-intercept by setting x to zero

Axis of symmetry is at average of roots

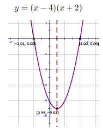
So x of vertex is there too!

Plug in x of vertex to get y of vertex.

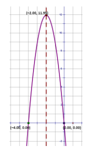
Graphing from *factored form*
> Graph the zeros
> Graph the axis of symmetry (midway between zeros)
> Find the vertex (plug x of the axis of symmetry into the function to find $f(x)$
> Sketch the curve. y-intercept is at apq

Try these. Sketch showing zeros, vertex, and axis of symmetry.

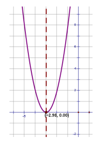
$y = (x - 4)(x + 2)$



$y = -3x(x + 4)$

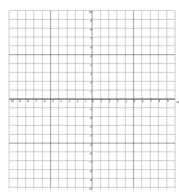


$y = 2(x + 3)^2$



Graphing from different forms:
Factored Form $y = a(x - p)(x - q)$
Vertex Form $y = a(x - h)^2 + k$
Standard form $y = ax^2 + bx + c$

Graph $f(x) = 2(x - 4)^2 - 3$



Vertex is easy to see

Look at a to visualize direction

y - intercept is easy (set x=0)

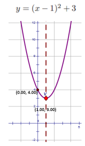
Include symmetric point

Find roots (take the square root)

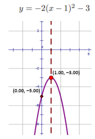
Graphing from *vertex form*
> Graph the vertex and axis of symmetry
> Use a to identify the direction and width (roughly)
> Find and graph the y-intercept ($y = ah^2 + k$) and it's symmetric point
> Find the zeros by taking square roots.
> Sketch the curve.

Try these. Sketch showing vertex, axis of symmetry, y-intercept, and approximate zeros.

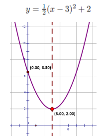
$y = (x - 1)^2 + 3$



$y = -2(x - 1)^2 - 3$

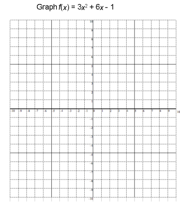


$y = \frac{1}{2}(x - 3)^2 + 2$



Graphing from different forms:
Factored Form $y = a(x - p)(x - q)$
Vertex Form $y = a(x - h)^2 + k$
Standard form $y = ax^2 + bx + c$

Graph $f(x) = 3x^2 + 6x - 1$



Find x of vertex (-b/2a)

Plug in to get y of vertex

Look at a to visualize direction

y - intercept is easy (it's c)

Include symmetric point

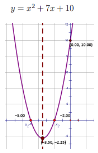
Find roots if needed

Graphing from *standard form*

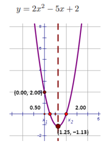
> Try factoring first. If possible, graph from factored form.
> Find and graph the axis of symmetry. $x = \frac{-b}{2a}$
> Find and graph the vertex (evaluate f at the axis of symmetry).
> Find and graph the y-intercept ($y = c$) and it's symmetric point.
> Use quadratic formula to find zeros if you really want accuracy.
> Sketch the curve. Test the approximate zeros in the function.

Try these. Sketch showing vertex, axis of symmetry, y-intercept, and zeros if factorable.

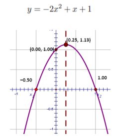
$y = x^2 + 7x + 10$



$y = 2x^2 - 5x + 2$



$y = -2x^2 + x + 1$



HW 6C.1 1bcf,2,3,4bef,5,6cfi,8beh
Do #7 for extra factoring practice!
QB 2, 3, 8

<u>Graphing "strategy"</u>

We've discussed graphing from different forms - what strategies help us do it ***efficiently***?

1. Completing the square
2. Using the discriminant

Completing the square to help graph a function

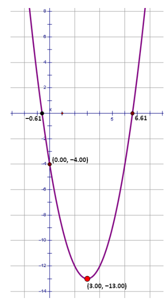
Suppose we want to graph the function $f(x) = x^2 + 6x - 4$

It would be nice to write it in the form $f(x) = a(x - h)^2 + k$ Why?
We can see the vertex and it's easier to graph!

So notice that $f(x) = x^2 + 6x - 4$ can be written as:
Add and subtract 9 to the RHS $f(x) = x^2 + 6x + 9 - 9 - 4$ which now has a "perfect square"
Rewrite with a binomial squared $f(x) = (x + 3)^2 - 13$ we now have "vertex form"

We can now graph the vertex at (-3, -13).
We can find the roots by finding the *x* values where the function is zero:

Set the function value to zero $0 = (x + 3)^2 - 13$
Add 13 to both sides $13 = (x + 3)^2$
Take the square root (don't forget $\pm$) $\pm\sqrt{13} = x + 3$
Subtract 3 and we're done! $x = -3 \pm \sqrt{13}$



It's also easy to find the *y*-intercept - it's -4
> From the original function in standard form or
> by plugging *x*=0 into the vertex form of the function

Completing the Square

A quadratic form of $x^2 + bx$ can be made into a perfect square binomial by adding $\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$

The resulting perfect square binomial is then $\left(x + \frac{b}{2}\right)^2$

Let's add a twist: What happens if the ***leading coefficient*** (*a*) is not one!

Short answer: Factor out *a*!

Consider this one $f(x) = 2x^2 - 16x + 7$
Factor out the leading coefficient $f(x) = 2[x^2 - 8x + \frac{7}{2}]$ Don't panic with fractions!
Add and subtract $\left(\frac{b}{2}\right)^2$ to complete the square $f(x) = 2[x^2 - 8x + 16 - 16 + \frac{7}{2}]$
Combine the left overs (fractions again) $f(x) = 2[(x - 4)^2 - \frac{32}{2} + \frac{7}{2}] = 2[(x - 4)^2 - \frac{25}{2}]$
Distribute the leading coefficient back through $f(x) = 2(x - 4)^2 - 25$

From here you can graph as we did above.

Another example

$f(x) = 3x^2 - 4x + 1$ in $a(x-h)^2 + k$

$3\left[x^2 - \frac{4}{3}x + \frac{4}{9} - \frac{4}{9} + \frac{3}{9}\right]$ $\frac{1}{3}\ \frac{3}{9}$ $x^2 + bx + \left(\frac{b}{2}\right)^2$

$3\left[\left(x - \frac{2}{3}\right)^2 - \frac{1}{9}\right]$

$3\left(x - \frac{2}{3}\right)^2 - \frac{1}{3}$

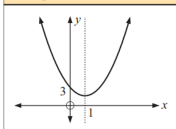
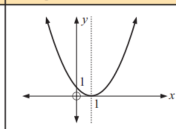
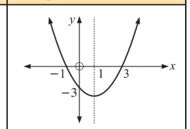
Recall the characteristics of the discriminant:

If $b^2 - 4ac > 0$, the radical is real. There are two real solutions.
If $b^2 - 4ac < 0$, the radical is imaginary. There are two complex solutions.

If $b^2 - 4ac = 0$, the radical is zero. There is one real solution at $x = \frac{-b}{2a}$.

If $b^2 - 4ac$ is a perfect square, you can factor the quadratic into integers!
If not, you can't!

IB uses the symbol Δ to represent the discriminant

$y = x^2 - 2x + 3$	$y = x^2 - 2x + 1$	$y = x^2 - 2x - 3$
		
$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(3)$ $= -8$	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(1)$ $= 0$	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(-3)$ $= 16$
$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
does not cut the x-axis	touches the x-axis	cuts the x-axis twice

If there is one solution, then Δ = 0. So we can answer questions like:

The function $f(x) = kx^2 - 8x + 2$ has a repeated root. Find the value of k.

Solution:

Since there is one root, $b^2 - 4ac = 0$

So $(-8)^2 - 4(k)(2) = 0$

or $64 - 8k = 0$ and $k = 8$

Or a more complex one: $f(x) = kx^2 - (k + 1)x - 3$ touches the x axis once. Find the value(s) of k.

Solution:

Since there is one root, $b^2 - 4ac = 0$

So $[-(k + 1)]^2 - 4(k)(-3) = 0$

or $k^2 + 2k + 1 - 12k = 0$ leading to $k^2 - 10k + 12 = 0$

Use quadratic formula or complete the square to solve $k^2 - 10k + 12 = 0$

$$k = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(12)}}{2(1)}$$

$$k == \frac{10 \pm \sqrt{100 - 48}}{2} = 5 \pm \frac{\sqrt{52}}{2} = 5 \pm \sqrt{13}$$

Positive Definite and Negative Definite

A quadratic is "Positive definite" if ***all*** of it's values are positive (not zero!).
Happens when: $a > 0$ (opens up) and discriminant is negative (no roots)

A quadratic is "Negative definite" if ***all*** of it's values are negative (not zero!).
Happens when: $a < 0$ (opens down) and discriminant is negative (no roots)

HW	6C.2	2bdf,3cf
	6C.3	1cf,2bd,3,4
	QB	6, 9, 10

Present: #1 all (verbally), #2, 3, 4

Positive Definite and Negative Definite

A quadratic is "Positive definite" if ***all*** of it's values are positive (not zero!).
Happens when: $a > 0$ (opens up) and discriminant is negative (no roots)

A quadratic is "Negative definite" if ***all*** of it's values are negative (not zero!).
Happens when: $a < 0$ (opens down) and discriminant is negative (no roots)

Finding Quadratics From a Graph

By understanding the features of a graph and how they relate to the equation, we can write an equation from a graph.

We need three pieces of information. Find the ones that are easiest to see (accurately), write an equation in an appropriate form, then track down the remaining ***parameters***.

Parameter

A parameter is a value in an equation, often a coefficient of the variable, that distinguishes the specific equation from others of the same form.

Examples:

Quadratic equations
- Factored Form $y = a(x - p)(x - q)$ Parameters are a, p, and q
- Vertex Form $y = a(x - h)^2 + k$ Parameters are a, h, and k
- Standard form $y = ax^2 + bx + c$ Parameters are a, b, and c

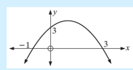
Linear equations
- Slope-intercept form $y = mx + b$ Parameters are m & b
- Point-slope form $y - y_1 = m(x - x_1)$ Parameters are m, x_1 & y_1
- Standard form $ax + by = c$ Parameters are a, b, and c

Exponential function
- $y = Ab^{p(x-q)} + d$ Parameters are A, b, p, q and d

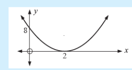
Examples:

Find the equation of the quadratic with graph:

a



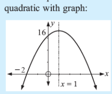
b



a Since the x-intercepts are -1 and 3,
$y = a(x+1)(x-3)$, $a < 0$.
But when $x = 0$, $y = 3$
$\therefore \quad 3 = a(1)(-3)$
$\therefore \quad a = -1$
So, $y = -(x+1)(x-3)$.

b Since it touches the x-axis at 2,
$y = a(x-2)^2$, $a > 0$.
But when $x = 0$, $y = 8$
$\therefore \quad 8 = a(-2)^2$
$\therefore \quad a = 2$
So, $y = 2(x-2)^2$.

Find the equation of the quadratic with graph:



The axis of symmetry is $x = 1$, so the other x-intercept is 4

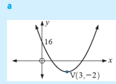
$\therefore \quad y = a(x+2)(x-4)$
But when $x = 0$, $y = 16$
$\therefore \quad 16 = a(2)(-4)$
$\therefore \quad a = -2$

$\therefore$ the quadratic is $y = -2(x+2)(x-4)$.

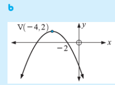
Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph cuts the x-axis at 4 and -3 and passes through the point $(2, -20)$.

Find the equation of the quadratic given its graph is:

a



b



HW 6D 1cf,2abc,3bdf,4def
QB 4,5,8,13(a-c)

Present 6D 1cf,2bc,3f,4f, QB 4,5,8,13(a-c)

E WHERE FUNCTIONS MEET

What does it mean for two functions to meet? They share a common (x, y) point!

In general, the intersections of ***any functions that you can graph*** can be found on a calculator!

Simpler cases can be done ***analytically***:

We will consider here only lines and parabolas. How many situations are there?

cutting (2 points of intersection) | **touching** (1 point of intersection) | **missing** (no points of intersection)

Finding the intersections is done by solving the ***system of equations***. (one linear, one quadratic)

Examples:

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$
$$\therefore\ x^2 - 2x - 15 = 0 \qquad \{\text{RHS} = 0\}$$
$$\therefore\ (x-5)(x+3) = 0 \qquad \{\text{factorising}\}$$
$$\therefore\ x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ the graphs meet at $(5, 2)$ and $(-3, -6)$.

When you get everything on one side you will have a quadratic with zero, one, or two solutions!
You can find out which by checking Δ!

$y = 2x + k$ is a tangent to $y = 2x^2 - 3x + 4$. Find k.

$y = 2x + k$ meets $y = 2x^2 - 3x + 4$ where

$$2x^2 - 3x + 4 = 2x + k$$
$$\therefore\ 2x^2 - 5x + (4 - k) = 0$$

Now this quadratic has $\Delta = 0$ since the graphs touch.

$$\therefore\ (-5)^2 - 4(2)(4-k) = 0$$
$$\therefore\ 25 - 8(4-k) = 0$$
$$\therefore\ 25 - 32 + 8k = 0$$
$$\therefore\ 8k = 7$$
$$\therefore\ k = \tfrac{7}{8}$$

Let's do one on a calculator to visualize:

A catapult launches a water balloon such that is follows the trajectory $h(t) = -16t^2 + 45 + 120$. A high powered rifle shoots a bullet along a more or less straight line described by $h(t) = 160 - 20t$.

a) At what time(s) are the two objects at the same height?
b) At what heights might the bullet pop the ballon?
c) What would the slope of the bullet line need to be to intercept the balloon path only once?

HW	6E	1bd,2cd,3d,5,7
QB	All except #7 by now!	

F PROBLEM SOLVING WITH QUADRATICS

Quadratics come up often in real world problems, often when optimizing something.

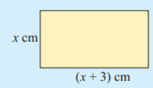
Success with problem solving takes practice - feeling into the territory and developing an instinct for what path will lead you to the solution. Some hints:

- Read the problem ***carefully***. Pay attention to detail and, in particular, what the question is asking for.
- Sketch a diagram, make a table or plot a graph.
- Define your variables carefully and clearly - ***write them down*** (including units of measure)
- Translate the problem into an equation ***that makes logical sense***.
- Do not try to write *x* = <blah> . When you do, you are essentially trying to solve the problem in your head.
- Check that the quantities in your equation are of the same units. You can only add and subtract like units.
- After you solve the equation, be sure to ***answer the question***. It is often not the same as the solution.

Here are some to try:

A rectangle has length 3 cm longer than its width. Its area is 42 cm^2. Find its width.

If the width is x cm then the length is $(x+3)$ cm.



$\therefore\ x(x+3)=42$ {equating areas}

$\therefore\ x^2+3x-42=0$

$\therefore\ x\approx -8.15$ or 5.15 {using technology}

We reject the negative solution as lengths are positive.

So, the width ≈ 5.15 cm.

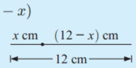
Is it possible to bend a 12 cm length of wire to form the legs of a right angled triangle with area 20 cm^2?

Suppose the wire is bent x cm from one end.

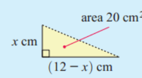
The area $A=\frac{1}{2}x(12-x)$

$\therefore\ \frac{1}{2}x(12-x)=20$

$\therefore\ x(12-x)=40$



becomes



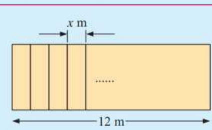
$\therefore\ 12x-x^2-40=0$

$\therefore\ x^2-12x+40=0$ which has $\Delta=(-12)^2-4(1)(40)$

$=-16$ which is <0

There are no real solutions, indicating this situation is **impossible**.

A wall is 12 m long. It is panelled using vertical sheets of timber which have equal width. If the sheets had been 0.2 m wider, 2 less sheets would have been required. What is the width of the timber panelling used?



Let x m be the width of each sheet.

$\therefore\ \frac{12}{x}$ is the number of sheets needed.

Now if the sheets had been $\left(x+\frac{1}{5}\right)$ m in width,

$\left(\frac{12}{x}-2\right)$ sheets would have been needed.

So, $\left(x+\frac{1}{5}\right)\left(\frac{12}{x}-2\right)=12$ {length of wall}

$\therefore\ 12-2x+\frac{12}{5x}-\frac{2}{5}=12$ {expanding LHS}

$\therefore\ -2x+\frac{12}{5x}-\frac{2}{5}=0$

$\therefore\ -10x^2+12-2x=0$ {$\times$ each term by $5x$}

$\therefore\ 5x^2+x-6=0$ {$\div$ each term by -2}

$\therefore\ (5x+6)(x-1)=0$

$\therefore\ x=-\frac{6}{5}$ or 1 where $x>0$

So, each sheet is 1 m wide.

HW 6F #1-19 all
QB #7 (*a* - *c*)

G QUADRATIC OPTIMISATION

Optimization problems involve finding the minimum or maximum of some value. Polynomials, particularly quadratics, often arise in such problems because they involve peaks and valleys.

Some hints:

- The ***vertex*** is the extreme point. Understand the meaning of it's *x* and *y* coordinates.
- The *x* coordinate is at -b/2a if the equation is in standard form.
- The sign of the leading coefficient will tell you whether the function has a minimum or a maximum.
- A function sometimes needs to be manipulated ***before*** looking for the vertex.
- A graphing calculator can be very helpful if it's allowed. See [2ND][CALC][MINIMUM] or [MAXIMUM]

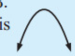
Here are some to try:

Find the maximum or minimum value of the following quadratics, and the corresponding value of x: **a** $y = x^2 + x - 3$ **b** $y = 3 + 3x - 2x^2$

a For $y = x^2 + x - 3$
$a = 1,\ \ b = 1,\ \ c = -3.$
As $a > 0$, the shape is 

$\therefore$ the minimum value occurs
when $x = \dfrac{-b}{2a} = -\frac{1}{2}$
and $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 3$
$= -3\frac{1}{4}$

The minimum value of y is $-3\frac{1}{4}$, occurring when $x = -\frac{1}{2}$.

b For $y = -2x^2 + 3x + 3$
$a = -2,\ \ b = 3,\ \ c = 3.$
As $a < 0$, the shape is 

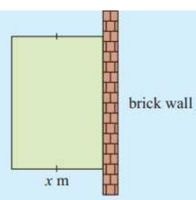
$\therefore$ the maximum value occurs
when $x = \dfrac{-b}{2a} = \frac{-3}{-4} = \frac{3}{4}$
and $y = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 3$
$= 4\frac{1}{8}$

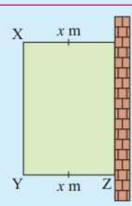
The maximum value of y is $4\frac{1}{8}$, occurring when $x = \frac{3}{4}$.

A vegetable gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. If the two equal sides are x m long:

a show that the area enclosed is given by $A = x(40 - 2x)$ m^2

b find the dimensions of the vegetable garden of maximum area.





a Side [XY] has length $(40 - 2x)$ m.
Now area = length $\times$ width
$\therefore$ $A = x(40 - 2x)$ m^2.

b $A = 40x - 2x^2 = -2x^2 + 40x$
is a quadratic in x, with $a = -2$, $b = 40$, $c = 0$.
As $a < 0$, the shape is 

The maximum area occurs when $x = \dfrac{-b}{2a} = \frac{-40}{-4} = 10$

$\therefore$ the area is maximised when YZ = 10 m and XY = 20 m.

HW 6G #1-11 all