

Chapter 5
Graphing and transforming functions

- A Families of functions
- B Transformation of graphs

A FAMILIES OF FUNCTIONS

Much of mathematics is based on just a few functions. There are general principals that apply to all functions. By understanding them, you can interpret, graph, and analyze an infinite number of common situations.

Some of the primary function families are given below. The parent function or (mother or father) is the simplest form, generally centered at the origin with unit coefficients. The family consists of variations of the parent function, obtained by changing various coefficients.

Name	General form	Parent
Linear	$f(x) = ax + b, a \neq 0$	$y = x$
Quadratic	$f(x) = ax^2 + bx + c, a \neq 0$	$f(x) = x^2$
Square Root	$f(x) = \sqrt{x} \quad x \geq 0$	$f(x) = \sqrt{x} \quad x \geq 0$
Cubic	$f(x) = ax^3 + bx^2 + cx + d, a \neq 0$	$f(x) = x^3$
Cube Root	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt[3]{x}$
Exponential	$f(x) = a^x, a > 0, a \neq 1$	$f(x) = b^x \quad b > 0$
Logarithmic	$f(x) = \log_e x$ or $f(x) = \ln x$	$f(x) = \log_b x \quad x > 0$
Reciprocal	$f(x) = \frac{k}{x}, x \neq 0$	$f(x) = \frac{1}{x} \quad x \neq 0$
Rational	$f(x) = \frac{ax+b}{cx+d} \quad x \neq -\frac{d}{c}$	$f(x) = \frac{1}{x} \quad x \neq 0$

It is important to know and recognize the major features of each function family. The general shape of each family should be familiar.

Family	Parent	Domain*	Range*	Asymp ?	Turn Pts. ?	X-Int*	Y-Int*
Linear	$y = x$	$x \in \mathbb{R}$	$y \in \mathbb{R}$	No	No	1	1
Quadratic	$f(x) = x^2$	$x \in \mathbb{R}$	$y \geq 0$	No	Yes	0, 1, 2	1
Square Root	$f(x) = \sqrt{x}$	$x \geq 0$	$y \geq 0$	No	No	1	1*
Cubic	$f(x) = x^3$	$x \in \mathbb{R}$	$y \in \mathbb{R}$	No	Yes	1, 2, 3	1
Cube Root	$f(x) = \sqrt[3]{x}$	$x \in \mathbb{R}$	$y \in \mathbb{R}$	No	No	1	1*
Exponential	$f(x) = b^x \quad b > 0$	$x \in \mathbb{R}$	$y > 0$	Yes	No	0*	1
Logarithmic	$f(x) = \log_b x$	$x > 0$	$y \in \mathbb{R}$	Yes	No	1	0*
Reciprocal	$f(x) = \frac{1}{x}$	$x \neq 0$	$y \neq 0$	Yes	No	0*	0*
Rational	$f(x) = \frac{ax+b}{cx+d} \quad x \neq -\frac{d}{c}$	$x \neq -\frac{d}{c}$	$y \neq -\frac{a}{c}$	H & V.!	No	0, 1	0, 1

*The domain, range and number of intercepts is for the parent function. A member of the family may have different values, depending on how it is shifted or stretched.

5 Rational functions

Let's look a bit further into reciprocal and rational functions.

The mother of all reciprocal and rational functions is ... $y = 1/x$

Let's look at it.

The Reciprocal Function

$$f(x) = 1/x$$

- The graph is a **hyperbola**
- Has a **vertical asymptote** at $x = 0$
- Has a **horizontal asymptote** at $y = 0$
- It is a **self-inverse** function

The General Rational Function

$$f(x) = \frac{p(x)}{q(x)}$$

- Is a ratio of **polynomials**
- Has **vertical asymptote(s)** where $q(x) = 0$
- Has **zeros** where $p(x) = 0$ (and $q(x) \neq 0$)
- May have horizontal or oblique asymptotes

The Linear Rational Function

$$f(x) = \frac{ax + b}{cx + d}$$

- The graph is a **hyperbola**
- Has a **vertical asymptote** at $x = -(d/c)$
- Has a **horizontal asymptote** at $y = a/c$

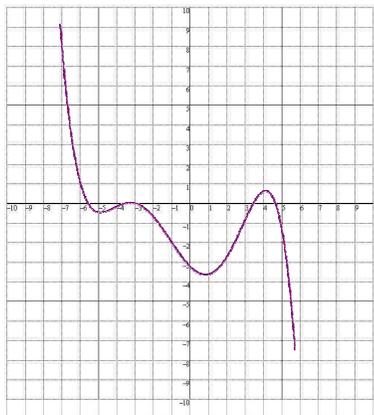
5A*: #2,4,5,7,9,10 (Function families)

Ox: Ch 5: #p. 133 #1-5 (Reciprocal & Rational)

B TRANSFORMATION OF GRAPHS

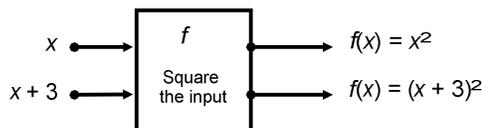
Starting from any function, $f(x)$, one can **transform** it in several ways. Graphically, the most common **transformations** are:

- Translations** - shift the graph horizontally or vertically by some amount
- Stretches** or compressions - scale the graph vertically or horizontally by some factor
- Reflections** - flip the graph over a given line
- Rotations** - rotate the graph around a given point by some angle.



Let's explore the changes in the graph and the changes in the related function.

Before you explore on your own, it's worth reviewing the idea of function composition. Suppose we have a function, f that squares whatever comes into it. We would write that as $f(x) = x^2$ since x represents "the thing we put into f ". To help talk about this, we use the word **argument** to represent the "expression that we put into a function". So in this case x is the argument of f . If we put $t + 5$ into f , the argument of f is $t + 5$ which is written $f(t + 5)$.



You will practice function composition more in the HW for section 5A.

For the remainder of this period, you will independently explore the relationship between the graph of a function, $f(x)$ and a **transformation** of f which we will call g that is given by $f(x - a) + b$

The exercises in section 5B.1 should help you understand this. I urge you to use your calculator, graphing both f and g on the same set of axes to see what happens.

Here's a GSP demo of the ideas.

Summary of *Translations*

- For $y = f(x) + b$, the effect of b is to **translate** the graph **vertically** through b units.
 - ▶ If $b > 0$ it moves **upwards**.
 - ▶ If $b < 0$ it moves **downwards**.
- For $y = f(x - a)$, the effect of a is to **translate** the graph **horizontally** through a units.
 - ▶ If $a > 0$ it moves to the **right**.
 - ▶ If $a < 0$ it moves to the **left**.
- For $y = f(x - a) + b$, the graph is translated horizontally a units and vertically b units. We say it is **translated by the vector** $\begin{pmatrix} a \\ b \end{pmatrix}$.

Translations are created by transforming a function $f(x)$ into $f(x - a) + b$

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Next, let's explore the effect of the transformations $f(x) = p \cdot f(x)$ and $f(x) = f\left(\frac{x}{p}\right)$

Notice that **dividing** x by p is the same as multiplying x by $\frac{1}{p}$

By doing the exercises in 5B.2 and paying attention to the patterns that you see, you should uncover the following properties of **dilations** (stretches and compressions).

Summary of *Dilations*

- For $y = pf(x)$, $p > 0$, the effect of p is to **vertically stretch** the graph by the factor p . p is called the **dilation factor**.
 - ▶ If $p > 1$ it moves points of $y = f(x)$ **further away** from the x -axis.
 - ▶ If $0 < p < 1$ it moves points of $y = f(x)$ **closer** to the x -axis.
- For $y = f\left(\frac{x}{q}\right)$, $q > 0$, the effect of q is to **horizontally stretch** the graph by a factor of q . q is called the **dilation factor**.
 - ▶ If $q > 1$ it moves points of $y = f(x)$ **further away** from the y -axis.
 - ▶ If $0 < q < 1$ it moves points of $y = f(x)$ **closer** to the y -axis.

The last transformations we'll explore involve $f(-x)$, $-f(x)$ (and for the ambitious, $-f(-x)$). Can you guess what the effect of these changes to the function will be? Go through the exercises in 5B.3 to explore.

Summary of *Reflections*

- For $y = -f(x)$, we **reflect** $y = f(x)$ **in the x -axis**.
- For $y = f(-x)$, we **reflect** $y = f(x)$ **in the y -axis**.

- 5B.1 1-4all,6a,7 Translations. (Use calculator or technology!)
- 5B.2 1cde,2ab,3,4ac,5ac,7,8no calc Stretches
- 5B.3 1ace,2,3,4,5,7 Reflections

There is no new information for section B.4. Just some more practice looking at a variety of transformations and combinations of transformations.

Practice in preparation for Quiz Tuesday

5B.4 1c,2c,3,4all
QB Practice #1-6
Review Sets 5A-C as needed

QB Problems

QB Mark Scheme