Solving exponential equations by equating exponents. First, the ***bases*** have to be the same!

$4^{2x+1} = 8^{1-x}$ $2^x \times 8^{1-x} = \frac{1}{4}$ $49^x + 1 = 2(7^x)$

GSP Exponentials

A LOGARITHMS

We have been looking at exponential functions of the form $f(x) = a \cdot b^{(cx-d)} + e$

A specific example might be the amount of money in a bank as a function of time:

$$A(t) = 1500(1.0125)^{4t}$$

We have explored certain questions all of which involved a known time t.

But what if we were to ask ***how long*** does it take for the amount to reach 2000?
We can start by writing the equation: $2000 = 1500(1.0125)^{4t}$

Dividing both sides by 1500 gives $\frac{4}{3} = (1.0125)^{4t}$

But this creates a troubling question: What do we have to raise 1.0125 to in order to get $\frac{4}{3}$? If we knew, we could divide that by 4 to get our answer.

We can frame the issue another way:
You know that $2^x = 8$ then x must be 3 since we know that 8 is 2 cubed.
Likewise, if $2^x = 16$ then x must be 4 since we know that 16 is 2^4

But...what would x be if $2^x = 10$?
It must be between 3 and 4 but what is it, exactly?

We call it the ***logarithm base 2 of 10*** or simply $\log_2(10)$

Definition of Logarithm

If $a^x = b$ then $\log_a b = x$

...or... $\log_a b$ is the *exponent* to which you raise the base a to get b.

Equivalent equations can be written in logarithmic or exponential form.

For example, $3^x = 12$ means $x = \log_3 12$

Try some: Write equivalent equations for the following:

$\log_3 9 = 2$ $3^2 = 9$	$\log_x 27 = 3$ $x^3 = 27$	$\log_4 12 = x$ $4^x = 12$
$4^2 = x$ $\log_4(x) = 2$	$3^t = 7$ $\log_3(7) = t$	$5^{(n-1)} = 2$ $\log_5(2) = n - 1$

Evaluate some logs for situations you know:

$\log_{10} 100\,000$

$\log_2 64$

$\log_2(0.125)$

$\log_3 243$

$\log_t \left(\frac{1}{t}\right)$

Now try to solve some equations:

$\log_3 x = 4$ $x = 81$	$\log_x 125 = 3$ $x = 5$	$\log_4(t - 24) = 5$ $x = 1048$

Notice a relationship between powers and logarithms: Consider $\log_b b^x$

This represents the power to which we raise b to get b^x. It's just x!

This can help simplify our lives: For example, $\log_3\left(\sqrt[5]{\frac{1}{81}}\right)$ is just $\log_3(3^{-4/5})$

So it evaluates to -4/5. Try a few:

a $\log_4 16$ 2	**b** $\log_2 4$ 2	**c** $\log_3\left(\frac{1}{3}\right)$ -1	**d** $\log_{10} \sqrt[4]{1000}$ 3/4

Calculating logs that are not known.
The ***log*** key on your calculator calculates log base 10 or the ***common logarithm***
The ***ln*** key on your calculator calculates log base e or the ***natural logarithm***
We will discuss how to find logs of other bases soon.

Another question: What is the $\log_2(-32)$

The domain of any log function must be restricted to positive arguments!

HW: 4A 1all,2all,3col4,4all,5all,6efgh

B LOGARITHMS IN BASE 10

Base 10 logs are particularly useful since we work in a decimal system.

Consider the following logs:

$\log_{10}(10)$	1	$\log_{10}(0.1)$	−1
$\log_{10}(100)$	2	$\log_{10}(0.01)$	−2
$\log_{10}(1000)$	3	$\log_{10}(0.001)$	−3
$\log_{10}(10^9)$	9	$\log_{10}(10^{-26})$	−26

Base 10 is so common that if the base is absent, it is assumed to be base 10!

So what would log(1235) be?

Your calculator will give you an exact answer, but you should know that it's between 3 and 4 (closer to 3). How about log(55,472)?

Let's work with these some more:

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 8 **b** 800 **c** 0.08

a $8 = 10^{\log 8} \approx 10^{0.9031}$

b $800 = 10^{\log 800} \approx 10^{2.9031}$

c $0.08 = 10^{\log 0.08} \approx 10^{-1.0969}$

a Use your calculator to find: **i** $\log 2$ **ii** $\log 20$

b Explain why $\log 20 = \log 2 + 1$.

a **i** $\log 2 \approx 0.3010$
ii $\log 20 \approx 1.3010$ {calculator}

b
$$\begin{aligned}\log 20 &= \log(2 \times 10)\\ &\approx \log(10^{0.3010} \times 10^1)\\ &\approx \log 10^{1.3010} \quad \{\text{adding indices}\}\\ &\approx 1.3010\\ &\approx \log 2 + 1\end{aligned}$$

Find x if:
a $\log x = 3$
b $\log x \approx -0.271$

a $x = 10^{\log x}$
$\therefore\ x = 10^3$
$\therefore\ x = 1000$

b $x = 10^{\log x}$
$\therefore\ x \approx 10^{-0.271}$
$\therefore\ x \approx 0.536$

HW: 4B 1dglp,2cdgh,3deij,4,5,6cdgk

C LAWS OF LOGARITHMS

Let's look at some properties of logarithms:

Let $\log(a) = x$ and $\log(b) = y$

Then, $10^x = a$ and $10^y = b$

Multiplying these two means $10^x \cdot 10^y = ab$

or $10^{x+y} = ab$

Using the definition, $\log(ab) = x + y$ or...

$$\log(ab) = \log(a) + \log(b)$$

Hmmm... think you could do somethings similar to find a rule for $\log(\frac{a}{b})$?

Let $\log(a) = x$ and $\log(b) = y$

Then, $10^x = a$ and $10^y = b$

Dividing these two means $10^x \div 10^y = \frac{a}{b}$

or $10^{x-y} = \frac{a}{b}$

Using the definition, $x - y = \log(\frac{a}{b})$ or...

$$\log(\frac{a}{b}) = \log(a) - \log(b)$$

Hmmm...another common situation is to evaluate $\log a^b$
Ideas?

Proof for the Power Rule

$\log_a x^n = n\log_a x$

Proof:

Step 1:
Let $m = \log_a x$

Step 2: Write in exponent form
$x = a^m$

Step 3: Raise both sides to the power of n
$x^n = (a^m)^n$

Step 4: Convert back to a logarithmic equation
$\log_a x^n = mn$

Step 5: Substitute for $m = \log_a x$
$\log_a x^n = n \log_a x$

$$\log a^b = b \cdot \log(a)$$

HW: 4C.1 1col3,2cfi,3ef,4ef,5becf,6becf,7all

Solving Equations with Logartihms

How do we solve equations that involve logarithms? First, let's look at how to write some equations in logarithmic form. As we do these, think about how we might work backward from the answer to the original equation.

Consider $y = 2^x$. Take the log base 10 of both sides to get $\log(y) = \log 2^x$ or $\log(y) = x\log 2$

Or how about $R = b\sqrt{l}$? Can you write it as a log base 10 equation? $\log(R) = \log(b) + 0.5\log(l)$

Try $L = \frac{ab}{c}$. $\log(L) = \log(a) + \log(b) - \log(c)$

> You can take the logarithm of both sides of an equation and use the rules of logarithms to rewrite an equation in logarithmic form.

Now try to try to ***remove*** the logarithms from some equations by using the rules of logarithms:

$\log D = \log e + \log 2$
$\log D = \log 2e$
$D = 2e$

$\log P = \frac{1}{2}\log x$
$\log P = \log x^{1/2}$
$P = x^{1/2} = \sqrt{x}$

$\log B = 3\log m - 2\log n$
$\log B = \log m^3 - \log n^2$
$\log B = \log \frac{m^3}{n^2}$
$B = \frac{m^3}{n^2}$

$\log P = 3\log x + 1$
$\log P = \log x^3 + \log 10$
$\log P = \log 10x^3$
$P = 10x^3$

Finally, let's use these ideas to ***solve*** some logarithmic equations:

$\log_3 27 + \log_3 \frac{1}{3} = \log_3 x$
$3 + (-1) = \log_3 x$
$2 = \log_3 x$
$9 = x$

$\log_5 125 - \log_5 \sqrt{5} = \log_5 x$
$3 - \frac{1}{2} = \log_5 x$
$\frac{5}{2} = \log_5 x$
$5^{\frac{5}{2}} = \sqrt{5^5} = 25\sqrt{5} = x$

$\log x + \log(x+1) = \log 30$
$\log x(x+1) = \log 30$
$x(x+1) = 30$
$x^2 + x - 30 = 0$
$(x+6)(x-5) = 0$
$x = 5$ or -6

> **Solving equations with logs: Strategy 1**
>
> 1) Manipulate both sides until there is one log term of the same base on each side.
> 2) Set the arguments of the two log functions equal.
> 3) Solve for the variable
> Note: Working with logarithmic equations can create ***extraneous solutions*** because the domain of the log function is restricted! Check your answers!

D NATURAL LOGARITHMS

Natural logarithms are logarithms with a base of the natural number e.
They follow the same rules as any other logarithm.
They are very common and $\log_e$ is written as ln.
Scientific calculators have a ln key on them

HW: 4C.2 1 col 3&4, 2 &3 last col
4D.1 1all,2,3,4all,5deij,6cdgh
Also QB 1, 2, 4, 5

Section D.2 is more practice with the laws of logarithms, this time using natural logs.

Do you recall how to work with equations that have constants in them?

Write the constant as a log of the base to the constant.

$$\ln M = 2\ln y + 3$$
$$\ln M = \ln y^2 + \ln e^3$$
$$\ln M = \ln e^3 y^2$$
$$M = e^3 y^2$$

$$\ln Q \approx 3\ln x + 2.159$$
$$\ln Q \approx \ln x^3 + \ln e^{2.159}$$
$$\ln Q \approx \ln e^{2.159} \cdot x^3$$
$$Q \approx e^{2.159} \cdot x^3 \approx 8.66x^3$$

E EXPONENTIAL EQUATIONS USING LOGARITHMS

We have looked at setting exponents equal to solve exponential equations.
But the bases on both sides have to be the same. When that is not possible, you can take the log (use any convenient base) of both sides of an equation.

Some examples:

$1.2^x = 1000$
$\log(1.2^x) = \log 1000$
$x \log 1.2 = 3$
$x = \dfrac{3}{\log 1.2} \approx 37.9$ (to 3 s.f.)

$12 \cdot 2^{-0.05t} = 0.12$
$2^{-0.05t} = 0.01$
$\log 2^{-0.05t} = \log 0.01$
$-0.05t \log 2 = -2$
$t = \dfrac{-2}{-0.05 \cdot \log 2} = \dfrac{40}{\log 2} \approx 133$ (3 s.f.)

Solving equations with exponents when the bases are different!

1) Simplify both sides as much as possible first.
2) Take the log of both sides, using an appropriate base depending on the situation.
3) Manipulate the equation with log rules until you can isolate the variable.
4) Simplify the result, using a calculator with proper rounding or as an exact answer.

HW: 4D.2 1-3, last col each, 4, 5bcefh
4E 1-3 last column of each
Also QB 6A, 8, 10, 13

F THE CHANGE OF BASE RULE

We know how to find logs if the base is 10 or *e*. What about other bases? Notice:

If $\log_b a = x$ then
$b^x = a$
$\log_c b^x = \log_c a$
$x \log_c b = \log_c a$
$x = \dfrac{\log_c a}{\log_c b} = \log_b a$

Change of Base Formula for Logarithms

$$\log_b a = \frac{\log_c a}{\log_c b}$$

Use whatever base is convenient for *c*.

Try a couple:

$\log_{\frac{1}{2}} 1250$ ≈ -10.3

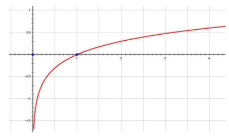
Solve for x: $8^x - 5(4^x) = 0$

$8^x - 5(4^x) = 0$
$\therefore\ 2^{3x} - 5(2^{2x}) = 0$
$\therefore\ 2^{2x}(2^x - 5) = 0$
$\therefore\ 2^x = 5$ $\{\text{as } 2^{2x} > 0 \text{ for all } x\}$
$\therefore\ x = \log_2 5$
$\therefore\ x = \dfrac{\log 5}{\log 2} \approx 2.32$

HW: 4F 1cd,2bc,3,4,5
Also QB 3,7,9abc,12,15

G GRAPHS OF LOGARITHMIC FUNCTIONS

What does a graph of a logarithm look like. Recall that logarithms are ***inverses of exponentials***.
Can you graph the function $f(x) = \log(x)$?



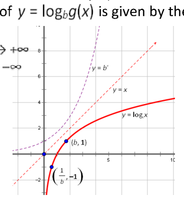
Let's look at these in more detail. GSP Exponentials & Logs

Properties of Graphs of Logarithms

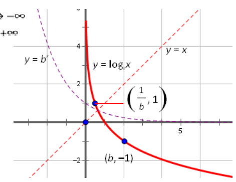
The graph of $y = \log_b(x)$ has the following properties for any base b:

> The curve is a reflection of a corresponding exponential across the line $y = x$ (they are inverses)
> The domain is $x > 0$ (we can only find logs of positive numbers)
> The graph has a vertical asymptote at $x = 0$ (the y-axis)
> The domain of $y = \log_b g(x)$ is given by the values of x that make $g(x) > 0$

> For $b > 1$:
as $x \to +\infty$, $y \to +\infty$
as $x \to 0^+$, $y \to -\infty$



> For $0 < b < 1$:
as $x \to +\infty$, $y \to -\infty$
as $x \to 0^+$, $y \to +\infty$



Consider the ***family*** of graphs that are generated by ***transforming*** $y = \log_b(x)$ by scaling and shifting the curve. The generalized form of the family of log functions is:

$$y = a\log_b(x - c) + d$$

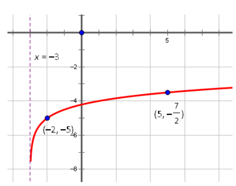
In a way that is analogous to exponential functions, we have:

Summary of Graphs of Logarithmic Curves

For $f(x) = a\log_b(x - c) + d$
- a stretches the curve vertically
  - $|a| > 0$ *stretches* the curve
  - $0 < |a| < 1$ *compresses* the curve
  - $a < 0$ *reflects over the x-axis* (flips vertically)
- b increases or decreases the rate of change (steepness) of the curve
  - $b > 1$ curve is *increasing*
  - $0 < b < 1$ curve is *decreasing*
  - b < 0 is for a future math class
- c shifts the curve *horizontally*
  - $c > 0$ shifts the curve to the right
  - $c < 0$ shifts the curve to the left
- d shifts the curve *vertically*
  - $d > 0$ shifts the curve *up*
  - $d < 0$ shifts the curve *down*

Try one. Sketch a graph of $f(x) = \frac{1}{2}\log_2(x + 3) - 5$

Hint: A good sketch will have at least 2 points labelled and show relevant asymptotes.



H GROWTH AND DECAY

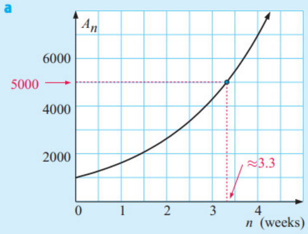
Very common application of exponents and logarithms involve real world growth and decay. Let's look at some examples:

A farmer monitoring an insect plague notices that the area affected by the insects is given by $A_n = 1000 \times 2^{0.7n}$ hectares, where n is the number of weeks after the initial observation.

a Draw an accurate graph of A_n against n and use your graph to estimate the time taken for the affected area to reach 5000 ha.

b Check your answer to **a** using logarithms and using suitable technology.

a



b When $A_n = 5000$,

$$1000 \times 2^{0.7n} = 5000$$
$$\therefore\ 2^{0.7n} = 5$$
$$\therefore\ \log 2^{0.7n} = \log 5$$
$$\therefore\ 0.7n \log 2 = \log 5$$
$$\therefore\ n = \frac{\log 5}{0.7 \times \log 2}$$
$$\therefore\ n \approx 3.32$$

$\therefore$ it takes about 3 weeks and 2 days.

Using technology we find the intersection of $y = 1000 \times 2^{0.7x}$ and $y = 5000$. This confirms $n \approx 3.32$.

Iryna has €5000 to invest in an account that pays 5.2% p.a. interest compounded annually. How long will it take for her investment to reach €20 000?

$u_{n+1} = 20\,000$ after n years

$u_1 = 5000$

$r = 105.2\% = 1.052$

Now $u_{n+1} = u_1 \times r^n$

$$\therefore\ 20\,000 = 5000 \times (1.052)^n$$
$$\therefore\ (1.052)^n = 4$$
$$\therefore\ \log(1.052)^n = \log 4$$
$$\therefore\ n \times \log 1.052 = \log 4$$
$$\therefore\ n = \frac{\log 4}{\log 1.052} \approx 27.3 \text{ years}$$

$\therefore$ it will take 28 years.

HW: 4G: #1bd,2bd,3,4,5,6
4H: #1,3,4-14 even
QB Practice #11,16