Chapter 3

Exponentials

A INDEX NOTATION

3^5 (base; power, index or exponent)

Thinking of this as 1 times 3, five times or (1)(3)(3)(3)(3)(3) may help you to understand negative exponents (later).

B EVALUATING POWERS

One of the biggest issues with powers is being clear what the base is.

Evaluate				
	$(-2)^3$	-2^3	$(-2)^4$	-2^4
	-8	-8	16	-16
	$(-3xy)^2$	$-3xy^2$	$-3(xy)^2$	$-(3xy)^2$
	$9x^2y^2$	$-3xy^2$	$-3x^2y^2$	$-9x^2y^2$

The exponent operates on the symbol that immediately precedes it!

C INDEX LAWS

Using your understanding of exponents, write down an equivalent power for each product.

$3^2 \cdot 3^2 = 3\cdot3 \cdot 3\cdot3 = 3^4$

$3^2 \cdot 3^3 = 3\cdot3 \cdot 3\cdot3\cdot3 = 3^5$

$3^2 \cdot 3^4 = 3\cdot3 \cdot 3\cdot3\cdot3\cdot3 = 3^6$

Pattern?

Product of power: $a^m \cdot a^n = a^{m+n}$

Try $5^2 \cdot 5^4$ $\quad -2(-2)^4$ $\quad x^2 \cdot x^3$

Using your understanding of exponents, write down an equivalent power for each power.

$(3^2)^2 = 3^2 \cdot 3^2 = 3\cdot3\cdot3\cdot3 = 3^4$

$(3^2)^3 = 3^2 \cdot 3^2 \cdot 3^2 = 3\cdot3\cdot3\cdot3\cdot3\cdot3 = 3^6$

Pattern?

Power of power: $(a^m)^n = a^{mn}$

Try $(4^2)^4$ $\quad [(-3)^5]$

Using your understanding of exponents, write down an equivalent power for each power.

$(3\cdot4)^2 = 3\cdot4 \cdot 3\cdot4 = 3\cdot3\cdot4\cdot4 = 3^2 4^2$

$(3\cdot4)^3 = 3\cdot4\cdot3\cdot4\cdot3\cdot4 = 3\cdot3\cdot3\cdot4\cdot4\cdot4 = 3^3 4^3$

Pattern?

Power of product: $(ab)^n = a^n b^n$

Try $(-3 \cdot 4)^2$ $\quad (2x)^2$ $\quad (-3x^2y)^3$

Can your understanding of exponents help you evaluate 2^{-4}?

$2^5 = 1\cdot2\cdot2\cdot2\cdot2\cdot2 = 32$ $\quad a^n$ means ***multiply*** 1 by a, n times

$2^4 = 1\cdot2\cdot2\cdot2\cdot2 = 16$

$2^3 = 1\cdot2\cdot2\cdot2 = 8$

$2^2 = 1\cdot2\cdot2 = 4$

$2^1 = 1\cdot2 = 2$

$2^0 = 1 = 1$

Zero Exponent Property: $a^0 = 1$ for all $a \neq 0$

$2^{-1} = \frac{1}{2} = \frac{1}{2}$

$2^{-2} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^2} = \frac{1}{4}$

$2^{-3} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^3} = \frac{1}{8}$

$2^{-n} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdots\cdot\frac{1}{2} = \frac{1}{2^n}$ $\quad a^{-n}$ means ***divide*** 1 by a, n times

Negative Exponent Property: $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$

This implies another useful form:

$a^n = \frac{1}{a^{-n}} = \left(\frac{1}{a}\right)^{-n}$

Try: $(2x)^{-3}$ $\quad 2xy^{-2}$ $\quad 2(xy)^{-2}$ $\quad \frac{1}{(2x)^{-3}}$ $\quad \frac{1}{2x^{-3}}$ $\quad \left(\frac{3}{4}\right)^{-n}$

Using your understanding of exponents, write down an equivalent power for each quotient.

$\frac{3^5}{3^5} = \frac{3\cdot3\cdot3\cdot3\cdot3}{3\cdot3\cdot3\cdot3\cdot3} = 1$

$\frac{3^5}{3^3} = \frac{3\cdot3\cdot3\cdot3\cdot3}{3\cdot3\cdot3} = 3\cdot3 = 3^2$

$\frac{3^5}{3^2} = \frac{3\cdot3\cdot3\cdot3\cdot3}{3\cdot3} = 3\cdot3\cdot3 = 3^3$

Pattern?

Quotient of powers: $\frac{a^m}{a^n} = a^{m-n}$; $\quad a \neq 0$

Try: $\frac{5^{17}}{5^{12}}$ $\quad \frac{-4^7}{-4^{12}}$ $\quad \frac{x^5}{x^3}$ $\quad \frac{x^5y^7}{x^7y^5}$

Using your understanding of exponents, write down an equivalent power for each power.

$\left(\frac{3}{4}\right)^5 = \frac{3\cdot3\cdot3\cdot3\cdot3}{4\cdot4\cdot4\cdot4\cdot4} = \frac{3^5}{4^5}$

$\left(\frac{4}{7}\right)^3 = \frac{4\cdot4\cdot4}{7\cdot7\cdot7} = \frac{4^3}{7^3}$

Pattern?

Power of a quotient: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$; $\quad b \neq 0$

Try: $\left(\frac{3}{4}\right)^3$ $\quad \left(\frac{x}{y}\right)^5$ $\quad \left(\frac{2x}{3y}\right)^{-4}$ $\quad \left(\frac{3x^2}{5y^{-3}}\right)^{-2}$

There are a lot of useful properties. ***Do not memorize them!*** Understand them!

Properties of Exponents

Let a and b be real numbers and m and n be integers. Then:

Product of Powers Property	$a^m \bullet a^n$	=	a^{m+n}
Power of a Power Property	$(a^m)^n$	=	a^{mn}
Power of a Product Property	$(ab)^m$	=	$a^m b^m$
Negative Exponent Property	a^{-m}	=	$\frac{1}{a^m} = \left(\frac{1}{a}\right)^m$, $a \neq 0$
Zero Exponent Property	a^0	=	1, $a \neq 0$
Quotient of Powers Property	$\frac{a^m}{a^n}$	=	a^{m-n}, $a \neq 0$
Power of a Quotient Property	$\left(\frac{a}{b}\right)^m$	=	$\left(\frac{a^m}{b^m}\right)$, $b \neq 0$

For now, we will restrict ourselves to ***integer*** exponents. More on that next time...

This takes practice...

Simplify using the index laws:

a $5^4 \times 5^7$ b $d^2 \times d^6$

g $\frac{p^3}{p^7}$ h $n^3 \times n^9$

Write as powers of 2:

a 4 b $\frac{1}{4}$

g 2 h $\frac{1}{2}$

Write as powers of 3:

a 9 b $\frac{1}{9}$

g 81 h $\frac{1}{81}$

Write as a single power of 2:

a 2×2^a b 4×2^b

f $\frac{2^c}{4}$ g $\frac{2^m}{2^{-m}}$

Write without negative indices:

a ab^{-2} b $(ab)^{-2}$

f $\frac{a^2 b^{-1}}{c^{-2}}$ g $\frac{1}{a^{-3}}$

Write in non-fractional form:

a $\frac{1}{a^n}$ b $\frac{1}{b^{-n}}$

Write without brackets:

a $(2a)^2$ b $(3b)^3$

f $\left(\frac{a}{3}\right)^3$ g $\left(\frac{b}{c}\right)^4$

Write the following in simplest form, without brackets:

a $(-2a)^2$ b $(-6b^2)^2$ c $(-2a)^3$

e $(-2ab^4)^4$ f $\left(\frac{-2a^2}{b^2}\right)^3$ g $\left(\frac{-4a^3}{b}\right)^2$

Simplify, giving answers in simplest rational form:

a $\left(\frac{5}{3}\right)^0$ b $\left(\frac{7}{4}\right)^{-1}$ c $\left(\frac{1}{6}\right)^{-1}$

e $\left(\frac{4}{3}\right)^{-2}$ f $2^1 + 2^{-1}$ g $\left(1\frac{2}{3}\right)^{-3}$

Write as powers of 2, 3 and/or 5:

a $\frac{1}{9}$ b $\frac{1}{16}$ c $\frac{1}{125}$

e $\frac{4}{27}$ f $\frac{2^c}{8 \times 9}$ g $\frac{9^k}{10}$

HW: 3C #1ijk,3ijk,4deij,7cdgh,8deij,9de,10cdgh,11cdgh

(Hint: It may be faster (and will certainly be better for you) to just do them all rather than looking at the above to see which ones are required!)

Some practice

Exponent Properties

Exponents > 0	$5^3 \cdot 5^4$	$x^2 \cdot x \cdot x^5$	$-4x^3 \cdot 3x^5$
	$(3^2)^3$	$(-x^3)^4$	$-2(3^2)^2$
	$(2x^4)^5$	$(-4n^3)^2$	$3(4k^5)^3$
	$3x(-x \cdot x^2)^3$	$2x^3 \cdot (-3x)^2$	$(abc^2)^3 \cdot ab$
	$\left(\frac{3}{4}\right)^2$	$\left(\frac{1}{x^{-3}}\right)^2$	$\left(\frac{m^2}{n}\right)^3$
Exponents < 0	$5^{-3} \cdot 5^4$	$2^0 \cdot 2^{-3}$	$n^{-2}(mn)^2$
	a^5b^{-8}	$a^6(a^3)^{-2}$	$4x^{-3}y^2z^{-5}$
	$3y^3x^{-4}$	$(6x^{-3})^3$	$(3^{-3})^2$
	$(2x)^{-3}$	$2x(yz)^{-2}$	$-2x(-yz)^{-2}$
	$\frac{3}{c^5}$	$\frac{1}{4^{-3}}$	$\frac{1}{(-6)^{-3}}$
	$\frac{4}{x^{-3}}$	$\frac{2y^{-2}}{x^3}$	$\frac{5y^2}{x^{-5}}$
	$\left(\frac{3}{4}\right)^{-2}$	$\left(\frac{1}{x^{-3}}\right)^{-2}$	$\left(\frac{m^2}{n}\right)^{-3}$

Divide Exponents	$\frac{6^5}{6^3}$	$\frac{(-5)^9}{(-5)^4}$	$\frac{(-t)^4}{-t}$
	$\frac{4^5}{4^{-2}}$	$\frac{(-3)^{-3}}{(-3)^{-6}}$	$\frac{n^{-4}}{n^5}$
	$\frac{x^5}{x^2}$	$\frac{16x^3y^2}{24x^2y^5}$	$\frac{4x^3}{y^2} \cdot \frac{2y^5}{x^5}$
	$\frac{16r^5s^9}{-2rs^2} \cdot \frac{r^2s}{-8}$	$\frac{4a^{-1}b^3}{a^4b^{-2}} \cdot \left(\frac{3a}{ab}\right)^{-2}$	$\frac{-6m^3n^{-2}}{5m^{-3}n^4} \cdot \frac{15m^2n}{-2n^3}$
Various Practice	$5^{-2} \cdot 5^2$	$(5^{-2})^{-2}$	x^{-5}
	$3^{-4} \cdot 3^7$	$4^7 \cdot 4^5$	$x^5 \cdot x^{-1}$
	$\left(\frac{3}{4}\right)^{-2}$	$\left(\frac{1}{n}\right)^{-3}$	$\left(-\frac{5}{16}\right)^0$
	$2x^{-3}$	$4t^{-1}$	$-7n^{-4}$
	$\frac{1}{6^3}$	$\frac{12}{y^4}$	$\frac{-1}{x^8}$
	10^{-3}	10^{-5}	10^{-1}
	$10^{-2} \cdot 10^8$	$10^{-1} \cdot 10^6$	$10^5 \cdot 10^2$
	$(-2)^2 \cdot (-2)^3$	$(-4)^7 \cdot (-4)^{-5}$	$(-15)^7 \cdot (-15)^4$
	$\left(\frac{3}{4}\right)^2 \cdot \left(\frac{3}{4}\right)^4$	$\left(\frac{4}{n}\right)^3 \cdot \left(\frac{4}{n}\right)^5$	$\left(-\frac{2}{3}\right)^5 \cdot \left(-\frac{2}{3}\right)^6$
		$5x^6 \cdot 9x^4$	$-2x^{-4} \cdot 8x^3$

D RATIONAL INDICES

We have discussed integer exponents. Considering the properties that we have developed, what *must* a fractional exponent represent? Consider, for example, the exponent of one half.

The product of powers property can help. It implies that $a^{\frac{1}{2}} \bullet a^{\frac{1}{2}} = a^{\left(\frac{1}{2}+\frac{1}{2}\right)} = a^1 = a$

But the number that multiplies by itself to get a already has a name: $\sqrt{a}$

So... $a^{\frac{1}{2}} = \sqrt{a}$ and in general $a^{\frac{1}{n}} = \sqrt[n]{a}$ which is the ***nth root of a***.

The properties that we have developed imply some other interesting things:

What happens if we cube the square root of 16? $\left(\sqrt{16}\right)^3 = (4)^3 = 64$

What happens if we take the square root of 16 cubed? $\sqrt{16^3} = \sqrt{4096} = 64$

Let's look at that using exponents:

$\left(16^{\frac{1}{2}}\right)^3 = 16^{\frac{3}{2}}$ means the (square root of 16) cubed...or the square root of (16 cubed)

In general,

Rational Exponents

$$a^{\frac{n}{m}} = \sqrt[m]{a^n} = \left(\sqrt[m]{a}\right)^n \qquad m \neq 0$$

Some practice: Write as single powers

$\frac{1}{\sqrt[4]{27}}$ $\frac{1}{\sqrt[5]{2}}$ $4^{-\frac{1}{2}}$ $8^{\frac{5}{3}}$ $\sqrt[4]{3}$ $(\sqrt{2})^3$

Try a couple on your calculator:

$4^{-\frac{3}{5}}$ $2^{\frac{7}{8}}$

Warm up: Simplify the expressions.

$\sqrt[3]{16} \cdot \sqrt[3]{4}$ 4

$(y^4)^{1/6}$ $y^{2/3}$

$\sqrt[3]{108} \cdot \sqrt[3]{4}$ $6\sqrt[3]{2}$

$\frac{\sqrt{3}}{\sqrt{75}}$ $\frac{1}{5}$

$\sqrt{\frac{20x^3y^2}{9xz^3}}$ $\frac{2xy\sqrt{5z}}{3z^2}$

$\left(\frac{7^3}{4^3}\right)^{-1/3}$ $\frac{4}{7}$

$(\sqrt[3]{-64})^4$ 256

$\frac{-3}{\sqrt[5]{x^6}}$ $\frac{-3\sqrt[5]{x^4}}{x^2}$

Simplify the expressions.

$(6^{2/3})^{1/2}$ $6^{1/3}$

$\sqrt[3]{\frac{x^{15}}{y^6}}$ $\frac{x^5}{y^2}$

$(16^{5/9} \cdot 5^{7/9})^{-3}$ $\frac{\sqrt[3]{50}}{16000}$

$\frac{x^{2/5}y}{xy^{-1/3}}$ $\frac{y^{4/3}}{x^{3/5}}$

$\frac{13^{3/7}}{13^{5/7}}$ $\frac{\sqrt[7]{371,293}}{13}$

$\frac{\sqrt[4]{36} \cdot \sqrt[4]{9}}{\sqrt[4]{4}}$ 3

$2\sqrt[6]{3} + 7\sqrt[6]{3}$ $9\sqrt[6]{3}$

$\sqrt[4]{12x^2y^6z^{12}}$ $yz^3\sqrt[4]{12x^2y^2}$

$6\sqrt[3]{5} + 4\sqrt[3]{625}$ $26\sqrt[3]{5}$

$(x^4y)^{1/2} + (xy^{1/4})^2$ $2x^2y^{1/2}$

E ALGEBRAIC EXPANSION AND FACTORISATION

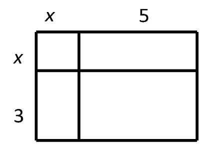
Remember algebra? Let's review a very important idea:

Multiplying binomials

$(a + b)(c + d)$

Various ways to remember it:
FOIL
Happy Man
Rainbow
Paperboy
Grid

$(x + 5)(x + 3)$



Try a few:

$(x - 6)(x + 2)$ $(3x - 8)(2x + 5)$ $(2x - 3y)(x - 4y)$

There are also some special patterns that you need to know to work quickly and efficiently. Understand where these come from and then ***memorize them***.

$$(a+b)(a-b) = a^2 - b^2$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

$(x - 6)(x - 6)$ $(x + 4)^2$ $(x - 7)^2$

In this context, it's all about doing the operations correctly:

$(2^x+1)(2^x+3)$ $2^{2x} + 4\cdot 2^x + 3$

$(2^x+3)(2^x-3)$ $2^{2x} - 9$ Aughhhhh!!!

$(3^x-1)^2$ $3^{2x} - 6^x + 1$

$(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)$ $x - 4$

$(7^x - 7^{-x})^2$ $7^{2x} - 2 + 7^{-2x}$

$(x + \frac{2}{x})^2$ $x^2 + 4 + \frac{4}{x^2}$

Also know how to deal with different situations:

$x^2(x^3 + 2x^2 + 1)$ $x^5 + 2x^4 + x^2$

$3^x(2 - 3^{-x})$ $2\cdot 3^x - 1$

$5^{-x}(5^{2x} + 5^x)$ $5^x + 1$

Practice is critical! Work thoroughly, make no move without absolutely knowing why. Also, be aware that next up is going in the other direction (aka factorisation). So watch the patterns.

HW is not a lot of ***practice***. Do more problems until you are getting them correct the first time - 1 minute per problem maximum!

HW 3E.1 #1cfi, #2cfil

Factoring

A huge skill, as important as dividing (it actually ***is*** dividing)

Factoring means writing an expression as a ***product of factors***.

In essence we are ***undoing*** multiplying. Look for ***common factors*** in each ***term***.

$6x + 18 = 6(x + 3)$

...but not just numbers! Common factors can be variables!
You may remember some like this: $3x^4 + 15x^3 - 6x^2 + 9x = 3x(x^3 + 5x^2 - 2x + 3)$

The same ideas hold with variables in the exponent:

$5^{2x} + 5^x = 5^x(5^x + 1)$ $3^{n+2} + 3^n = 3^n(3^2 + 1) = 10 \cdot 3^n$

You may be able to factor an expression even if it has no common factors!
There are several methods to try:

1) Reverse FOIL
   - Guess and check
   - The British method (for quadratics with ***a*** not equal to one)
2) Recognizing a special pattern
3) Factor by grouping

Let's review

Special Factoring Patterns

Squares of binomials	Cubes of binomials
$a^2 + 2ab + b^2 = (a + b)^2$	$a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$
$a^2 - 2ab + b^2 = (a - b)^2$	$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$
Difference of two squares	**Sum of two cubes**
$a^2 - b^2 = (a + b)(a - b)$	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
	Difference of two cubes
	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$b^2 + 3b - 40 = (b + 5)(b - 8)$ $a^2 + 12a - 28 = (a + 14)(a - 2)$

$21x^2 - 77x - 28 = 7(3x + 1)(x - 4)$ $10x^2 + 55x - 105 = 5(2x - 3)(x + 7)$

$9x^2y^4 - 169w^2 = (3xy^2 + 13w)\cdot(3xy^2 - 13w)$ $64x^8 - 81 = (8x^4 + 9)\cdot(8x^4 - 9)$

$49t^2 + 70t + 25 = (7x + 5)^2$ $36x^2 - 24xy + 4y^2 = (6x - 2y)^2$

$12x^3 - 20x^2 - 27x + 45 = (3x - 5)(2x + 3)(2x - 3)$

$12x^3 + 14x^2 + 26x + 18 = (4x + 2)(3x^2 - 5x + 9)$

In SL, these ideas get extended in many directions, including with exponents:

Factorise:

a $9^x - 4$ $(3^x + 2)(3^x - 2)$

d $25 - 4^x$ $(5 + 2^x)(5 - 2^x)$

g $9^x + 10(3^x) + 25$ $(3^x + 5)(3^x - 1)$

Factorise:

a $4^x + 9(2^x) + 18$ $(2^x + 6)(2^x + 3)$

d $9^x + 4(3^x) - 5$ $(3^x - 1)(3^x + 5)$

Simplifying - Writing an expression in an equivalent form

We also need to know how to ***simplify*** expressions involving exponents:

Simplify:

a $\frac{12^n}{6^n}$ 2^n

e $\frac{35^x}{7^x}$ 5^x

Simplify:

a $\frac{6^m + 2^m}{2^m}$ $3^m + 1$

d $\frac{12^x - 3^x}{3^x}$ $4^x - 1$

g $\frac{5^{n+1} - 5^n}{5^n}$ 4

Simplify:

a $2^n(n + 1) + 2^n(n - 1)$ $n \cdot 2^{n+1}$

HW 3E.2 1cf,2cfi,3cef,4cdgh,5cfi,6

F EXPONENTIAL EQUATIONS

Exponential functions often occur in equations that we need/want to ***solve.***

Suppose you invest $100 at an annual interest rate of 5% compounded annually.
How many years will it take for your investment to reach $200.

Remember the equation? $u_{n+1} = u_1(1+r)^n$

In this case we start with $100 and want to end up with $200. So we're looking at

$$200 = 100(1+0.05)^n \text{ or } 2 = 1.05^n$$

We've seen this a little bit before. In general, we need to use logs. But we'll save that for the next chapter. Meanwhile, notice that there are some forms of exponential equations that we can solve without using logs. Suppose, for example, that we wanted to solve the equation $8 = 2^n$

Because this can be written as $2^3 = 2^n$ we can mathematically support what we know to be true.

Equating Exponents

If you can rewrite an exponential equation to have powers of the same base on either side, you can ***equate the exponents*** to solve the equation.

Try a few

Solve for x: **a** $2^x = 16$ **b** $3^{x+2} = \frac{1}{27}$

a $2^x = 16$
$\therefore\ 2^x = 2^4$
$\therefore\ x = 4$

b $3^{x+2} = \frac{1}{27}$
$\therefore\ 3^{x+2} = 3^{-3}$
$\therefore\ x+2 = -3$
$\therefore\ x = -5$

Solve for x:
a $4^x = 8$
b $9^{x-2} = \frac{1}{3}$

a $4^x = 8$
$\therefore\ (2^2)^x = 2^3$
$\therefore\ 2^{2x} = 2^3$
$\therefore\ 2x = 3$
$\therefore\ x = \frac{3}{2}$

b $9^{x-2} = \frac{1}{3}$
$\therefore\ (3^2)^{x-2} = 3^{-1}$
$\therefore\ 3^{2(x-2)} = 3^{-1}$
$\therefore\ 2x-4 = -1$
$\therefore\ 2x = 3$
$\therefore\ x = \frac{3}{2}$

Another idea: Notice the quadratic form. Factor and ***then*** find each zero.

Solve for x: $4^x + 2^x - 20 = 0$

$4^x + 2^x - 20 = 0$
$\therefore\ (2^x)^2 + 2^x - 20 = 0$ {compare $a^2 + a - 20 = 0$}
$\therefore\ (2^x - 4)(2^x + 5) = 0$ {as $a^2 + a - 20 = (a-4)(a+5)$}
$\therefore\ 2^x = 4$ or $2^x = -5$
$\therefore\ 2^x = 2^2$ {2^x cannot be negative}
$\therefore\ x = 2$

A few more. Notice how you can manipulate things. You're looking for:
> One term on each side
> The same base on each side (rewrite the bases)
> Expressions in the exponents that you can set equal to each other
> Also look for quadratic forms

Solve for x:
a $2^x = 8$ — **a** $x = 3$
e $3^x = \frac{1}{3}$ — **e** $x = -1$
i $2^{x-2} = \frac{1}{32}$ — **i** $x = -3$

Solve for x:
a $8^x = 32$ — **a** $x = \frac{5}{3}$
e $27^x = \frac{1}{9}$ — **e** $x = -\frac{2}{3}$
i $4^{4x-1} = \frac{1}{2}$ — **i** $x = \frac{1}{8}$
m $81^x = 27^{-x}$ — **m** $x = 0$

Solve for x:
a $4^{2x+1} = 8^{1-x}$ — **a** $x = \frac{1}{7}$

Solve for x:
a $3 \times 2^x = 24$ — **a** $x = 3$
d $12 \times 3^{-x} = \frac{4}{3}$ — **d** $x = 2$

Solve for x:
a $4^x - 6(2^x) + 8 = 0$ — **a** $x = 1$ or 2
d $9^x = 3^x + 6$ — **d** $x = 1$

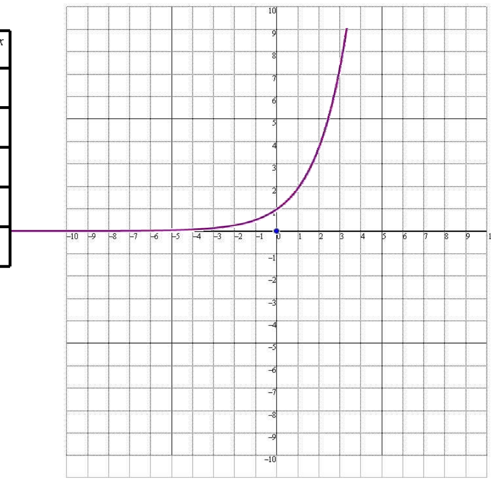
This is the ***minimum*** homework. Do enough to become proficient.

HW 3F: 1dhi,2dhlp,3c,4cf,5cf

G GRAPHS OF EXPONENTIAL FUNCTIONS

Graphs help us to visualize relationships (functions). What does an exponential look like? Start with a table of values:

x	$y = 2^x$
0	1
1	2
2	4
-1	1/2
-2	1/4



Notice that the function can never reach zero. It has an ***asymptote*** at *y*=0.
What is the range of the function? {y|*y*>0}
What is the affect of changing the base?
How about the ***parameter*** *a*?

GSP Demo

The general exponential has four parameters that control its shape:

For the general exponential function $y = a \times b^{x-c} + d$

- b controls how steeply the graph increases or decreases
- c controls horizontal translation
- d controls vertical translation and $y = d$ is the equation of the horizontal asymptote.
- ▸ if $a > 0$, $b > 1$ the function is increasing. 
  ▸ if $a > 0$, $0 < b < 1$ the function is decreasing. 
  ▸ if $a < 0$, $b > 1$ the function is decreasing. 
  ▸ if $a < 0$, $0 < b < 1$ the function is increasing. 

Understanding this, we can graph exponential functions.

To graph:
> Plot the *y*-intercept
> Plot two other points (*x* = 1 and *x* = -1 are usually easy)
> Know the general shape and connect

To find the function from a graph is not so easy because there are so many possibilities.

HW 3G: 1bdf,2all,4-6

H GROWTH AND DECAY

Imagine that 100 rabbits are dropped onto an island and they immediately start reproducing at the rate of 50% per year. Can you write a function that gives the population of rabbits at some number of years *n* after the initial drop off? Can you graph the function?

This phenomenon can be modeled by a ***geometric sequence*** because the number of rabbits in year *n* is a constant multiple (1.5) of the number of rabbits in the previous year. (Assume no rabbits die!)

Recall from our work with geometric sequences that the number of rabbits born in year *n* + 1 is:

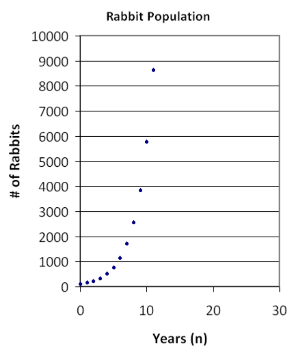
$$u_{n+1} = u_1(1 + r)^n$$

where *r* is the annual percentage growth rate written as a decimal. Another common form of this formula uses 0 as the first index and is written as: $u_n = u_0(1 + r)^n$

So the function that describes the population after the n^{th} year is: $P(n) = 100(1.5)^n$

Let's look at the graph of this function:

	r: 50.00%
n	u_n
0	100
1	150
2	225
3	337.5
4	506.25
5	759.375
6	1139.063
7	1708.594
8	2562.891
9	3844.336
10	5766.504
11	8649.756
12	12974.63
13	19461.95
14	29192.93
15	43789.39
16	65684.08
17	98526.13
18	147789.2
19	221683.8
20	332525.7



Wow - This is the equivalent of 3 child families (assuming no deaths, of course!) But you get the picture. More realistic: World population growth rate is approximately 1.17% per year:

	r: 1.17%
n	u_n
0	6990000000
1	7071783000
2	7154522861
3	7238230779
4	7322918079
5	7408596220
6	7495276796
7	7582971534
8	7671692301
9	7761451101
10	7852260079
11	7944131522
12	8037077861
13	8131111672
14	8226245679
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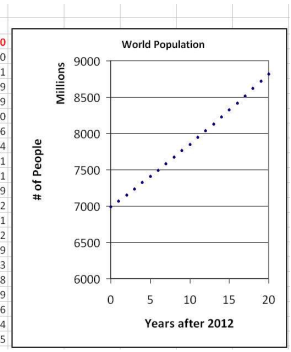
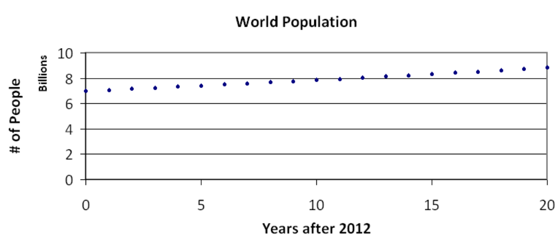
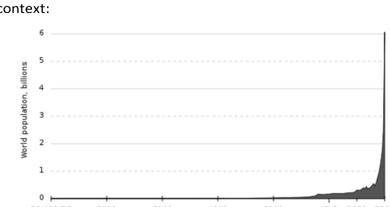


Here's the same data in another view. What does it communicate?



Consider this context:



So ask yourself why some understanding of math is important....

Exponential functions also describe ***decay***. A common example is radioactive decay where, for example, a radioactive substance deteriorates (decays) at a rate of 5% per year. Now we have a "growth multiplier" (1 + *r*) that is less than 1. It's really (1 - *r*).

The amount of the substance after *n* years is given by: $W_n = W_0(0.95)^n$

Suppose we started with 50 grams. We can ask questions such as:

> How much is present after 10 years?
> When will the amount left be less than 10 grams? (technology will be helpful here)

Radioactive materials are often described by their half-life. This is the amount of time it takes to reach half the initial weight. Given that some material has a half life of 1000 years, what is the annual rate of decay?

The exponent is not always an integer number like *n*. Let's look at some examples:

The weight of radioactive material remaining after t years is given by $W_t = W_0 \times 2^{-0.001t}$ grams.

a Find the original weight.

b Find the percentage remaining after 200 years.

a When $t = 0$, $W_0 = W_0 \times 2^0 = W_0$
$\therefore$ W_0 is the original weight.

b When $t = 200$,
$$\begin{aligned} W_{200} &= W_0 \times 2^{-0.001 \times 200} \\ &= W_0 \times 2^{-0.2} \\ &\approx W_0 \times 0.8706 \\ &\approx 87.06\% \text{ of } W_0 \end{aligned}$$
$\therefore$ 87.1% remains.

SNOWMOBILE The value of a snowmobile has been decreasing by 7% each year since it was new. After 3 years, the value is $3000. Find the original cost of the snowmobile.

$3,729.69

HW 3H.1 all
3H.2 all

I THE NATURAL EXPONENTIAL 'e'

Let's go back to compound interest again. You will recall that the formula for the amount of money in the bank after *t* years if you invest *P* dollars at an annual interest rate of *r*, compounded annually is given by:

$$A = P(1 + r)^t \text{ compounded annually}$$

If you compound, say, quarterly, you will be paid 4 times as often but the interest rate you receive each quarter is only a fourth of the annual interest rate. Thus, the formula becomes:

$$A = P(1 + \frac{r}{4})^{4t} \text{ compounded quarterly}$$

If you compound monthly, there are 12 periods. So the formula becomes:

$$A = P(1 + \frac{r}{12})^{12t} \text{ compounded monthly}$$

A question arises: what if you compounded "continuously" meaning that we let the number of periods in year get larger and larger - on to infinity! Let's look and see if there's a pattern. To do this, we need to generalize our equation first. Let's use *n* to represent the number of compoundings. Then we have:

$$A = P(1 + \frac{r}{n})^{nt} \text{ compounded } n \text{ times/year}$$

To explore this, let's make a substitution $a = \frac{n}{r}$. Show algebraically that $A = P[(1 + \frac{1}{a})^a]^{rt}$

This is nice because I can explore the factor in square brackets. Notice that as *n* gets larger and larger, so does *a* (it's just *n* divided by a constant). So let's see what happens as *a* gets large.

a	$(1 + \frac{1}{a})^a$
1	2
10	2.593742
100	2.704814
1000	2.716924
10000	2.718146
100000	2.718268
1000000	2.71828
10000000	2.718282
100000000	2.718282

This number is the ***natural number*** called "***e***" after Leonhard Euler who discovered it. It shows up in nature in many strange and interesting ways.

It also is the sum of $1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \cdots$

It is irrational!

HW 3I: #1,3,47cd,8cd,9dh,10,12,14
Review Set 3C p. 118