A NUMBER PATTERNS

A ***number sequence*** is any list of numbers with a recognizable pattern.

The members of a sequence are called ***terms***.

The terms are usually given a single name like u or x or t with a subscript that indicates the position in the sequence.

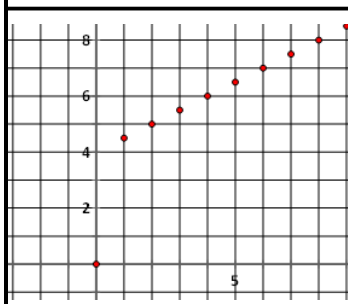
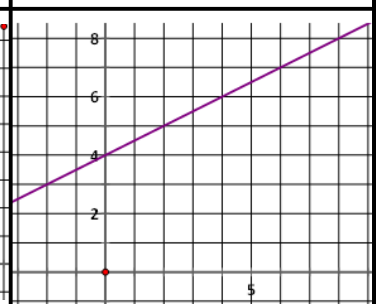
The subscript is called the ***index***. Sometimes the index starts at 0, other times at 1. If it's not stated, we usually assume that the index starts at 1. We generally use the letter ***n*** to represent a general term. (Note: n sometimes also refers to the total number of terms in a sequence. Watch the context.)

u_n is the n^{th} term in a sequence of u values.

Sequences can be represented with words, lists, formulas, or graphs.

The notation $\{u_n\}$ represents a function that generates the sequence defined by using u_n as the n^{th} term.

Example:
$\{0.5n + 4\}$ generates the sequence 7, 10, 13, 16, 19...
Note how this differs from a continuous function:

Sequence	**Continuous**
$\{u^n\} = \{3n + 4\}$	$f(x) = 3x + 4$
Domain: n an integer > 0	Domain: x a real number
	

C ARITHMETIC SEQUENCES

An ***Arithmetic Sequence*** is a sequence in which each term differs from the previous one by the same amount.

Algebraically:

$\{u_n\}$ is ***arithmetic*** if and only if $u_{n+1} = u_n + d$ for all integers $n > 0$

The difference between each term, ***d***, is called the ***common difference***.

Some features of arithmetic sequences:

The second term of any three successive terms is the arithmetic mean of the first and third terms. (Can you prove that?)

The value of the n^{th} term is given by $u_1 + (n - 1)d$
(Can you prove that?)

2A: 1, 2cfi, 3, 4def
2B: 1dh, 2, 3
2C: 2, 3, 5, 7def, 8cd, 9a, 11

D GEOMETRIC SEQUENCES

A ***Geometric Sequence*** is a sequence in which the ratio of successive terms is constant.

Algebraically:

$\{u_n\}$ is ***geometric*** if and only if $\frac{u_{n+1}}{u_n} = r$ for all integers $n > 0$

The ratio of successive terms, ***r***, is called the ***common ratio***.

Some features of geometric sequences:

They increase or decrease ***exponentially***

There can be no zero terms! (Do you know why?)

Can you write the general expression for the n^{th} term?

Consider a geometric sequence with a first term of 3 and a common ratio of 2.

a) Write out the first five terms *without performing any multiplications*

b) See if you see a pattern that enables you to write the general term *n*.

c) Generalize using *u*, *r*, and *n*.

n = 1, 2, 3, 4, 5 n

$u_1, ru_1, r(ru_1), r(r(ru_1)), r(r(r(ru_1))), \ldots. r(r(\ldots(r(ru_1)))$

$u_1, ru_1, r^2u_1, r^3u_1, r^4u_1, \ldots. r^{(n-1)}u_1$

The general expression for the n^{th} term of a geometric sequence is:

$$u_n = r^{(n-1)}u_1$$

Applications of geometric sequences

Imagine that you have \$100 in a bank and that they pay you 3% per year. How much do you have in the bank after n years?

To understand this more easily, notice that ***adding*** 3% to something is the same as multiplying it by 1.03.

$a + .03a = a(1 + .03) = a(1.03) = 1.03a$

So let's make a table of values:

Year	Calculation	Balance
0		\$100.00
1	(1.03)(100)	\$103.00
2	(1.03)(1.03)(100)	\$106.09
3	(1.03)(1.03)(1.03)(100)	\$109.27
4	(1.03)(1.03)(1.03)(1.03)(100)	\$112.55
...	...	...
n	$(1.03)^n(100)$	

Compound Interest (annual compounding)
If you invest u_1 dollars for n years at an interest rate of r % per year the value after n years is given by

$$u_{n+1} = u_1 (1 + r)^n$$

Note: The book uses a slightly different formula. Some formulas define r as the interest rate. Others define r as a "growth multiplier" or a "growth rate" in which the "1 +" is already included. Pay attention!

Periodic Compounding : If interest is compounded over a different time period, you need to adjust r and n. Understand that r is the interest rate earned *each period*. So if the annual rate is 5% and the compounding is every month, the *rate per period* is 5%/12 or 0.41667% (=.0041667). Each year represents 12 periods. So in this case, if you started with P dollars after *n years* you would have:

$$A = P(1 + .004667)^{12n}$$

12 *months* each year

Annual rate divided by 12

Another example

Desert Academy's current enrollment is 175 (more or less). Suppose that it increases by 4% per year.

What would the enrollment be after 5 years?
In what year would we exceed our maximum capacity of 250?

Very similar to interest:

$$\textit{Value after n years} = u_1 (1 + r)^n$$
$$= 175(1.04)^5 = 212.91 \cong 213 \text{ students}$$

What about the second question?
Can you write the equation we are trying to solve?

$$250 \leq 175(1.04)^n$$

For now, use trial and error or graphing. More on this problem later!

2D.1 1,3bcd,4,7,8,9c,10bc
2D.2 3,5,6,8,10
2D.3 1,2

E SERIES

A ***series*** is a ***sum*** of the terms in a sequence.

$S_n = u_1 + u_2 + u_3 + \ldots + u_n$

Sigma Notation: We use the Greek letter ***sigma*** to represent sums as follows:

$$S_n = \sum_{i=1}^{n} u_i$$

n = the index of the first term in the series

u_i = an expression defining the terms being added

i = the letter indicates which letter in the expression is the index
1 = the first number to use for the index in the series

Note: We use S_n to represent ***series*** and u_n to represent ***sequences***.

Write out a few:

$$\sum_{i=1}^{4} \frac{1}{i} \qquad \sum_{i=1}^{5} i \qquad \sum_{k=-4}^{4} (2k+3)^{k-1}$$

Properties of sigma notation

$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ sums can be broken apart

$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$ constants, *c*, can be factored out

$\sum_{i=1}^{n} c = cn$ summing a constant is like multiplying

2E.1 1def,2cd,3,4

Arithmetic Series (the sum of an arithmetic sequence)

Consider the arithmetic series $S_n = 12 + 15 + 18 + \dots + 529 + 531 + 534$

How many terms are in it? $\frac{534 - 12}{3} = \frac{522}{3} = 174 + 1 = 175$

What is the sum of the terms?

If you start with the original...	$S_n = 12 + 15 + 18 + \dots + 529 + 531 + 534$
rewrite it in the opposite order	$S_n = 534 + 531 + 529 + \dots + 18 + 15 + 12$
Add them to get twice the sum.	$2S_n = 546 + 546 + 546 + \dots + 546 + 546 + 546$
We know the number of terms	175 terms
So we can get the sum	$2S_n = 175 \cdot 546$ so the sum is half that = 47775

Notice that 546 is the sum of the first and last terms. To get the sum of the series, just multiply that by n and divide by 2. This is true in general!

Properties of Arithmetic Series

Given u_1 and the common difference d:
The n^{th} term (recall from sequences):

$$u_n = u_1 + d(n-1)$$

The sum of n terms:

$$S_n = \frac{n}{2}(u_1 + u_n) \text{ or } S_n = \frac{n}{2}(2u_1 + d(n-1))$$

These tend to be key ideas in Type I internal assessments!

Can you write the above as an equation using sigma notation?

$$S_n = \sum_{k=1}^{n} u_1 + d(k-1) = \frac{n}{2}(u_1 + u_n)$$

Geometric Series (the sum of a geometric sequence)

What is the sum of the terms?

Start with a general series	$S_n = u_1 + u_1r + u_1r^2 + \ldots + u_1r^{n-2} + u_1r^{n-1}$
Multiply both sides by r (this increases each power of r by 1)	$rS_n = u_1r + u_1r^2 + u_1r^3 + \ldots + u_1r^{n-1} + u_1r^n$
	This much is the sum, minus the first term
Condense RHS	$rS_n = S_n - u_1 + u_1r^n$
A little algebra...	$rS_n - S_n = -u_1 + u_1r^n$
will take us to...	$S_n(r-1) = u_1(r^n - 1)$
the result we want:	$S_n = \frac{u_1(r^n-1)}{(r-1)}$ or equivalently $S_n = \frac{u_1(1-r^n)}{(1-r)}$

Properties of Geometric Series

Given u_1 and the common ratio $r \neq 1$:
The n^{th} term (recall from sequences):

$$u_n = r^{(n-1)}u_1$$

The sum of n terms:

$$S_n = \frac{u_1(r^n-1)}{(r-1)} \text{ or equivalently } S_n = \frac{u_1(1-r^n)}{(1-r)}$$

These tend to be key ideas in Type I internal assessments!

Can you write the above as an equation using sigma notation?

$$S_n = \sum_{k=0}^{n-1} u_1 r^k = u_1 \sum_{k=0}^{n-1} r^k = \frac{u_1(1-r^n)}{(1-r)} = \frac{u_1(r^n-1)}{(r-1)}$$

Think about what this formula will do as the number of terms increases.

If $r > 1$, each successive term gets bigger and the sum will ***diverge***. (see numerator)

But if $r < 1$, each successive term gets smaller. What happens as $n \to \infty$

The series will ***converge*** to $\frac{u_1}{(1-r)}$

Sum of an Infinite Geometric Series

The ***limiting sum*** of an infinite series with $|r|<1$ as $n \to \infty$ is given by: $S_\infty = \frac{u_1}{(1-r)}$

What happens when $r < 0$. Hmmmm....

These are called ***alternating series***.

2E.2 1d,2bc,3,**4-12 even** (change from plan)
2E.3 1cd,2cd,3,4c,6
2E.4 2,3,5,6,7