Algebra and Functions

- Algebra - 8 hours
  - Sequences and series
  - Exponentials and logarithms
  - Binomial theorem (if time)
- Functions - 24 hours
  - Concepts
  - Graphs
  - Transformations
  - Graphs and equations of function families
    - Linear
    - Quadratic (and other polynomial)
    - Reciprocal
    - Exponential
    - Logarithmic

See syllabus for details
Finish just before Easter
End the year with Trig

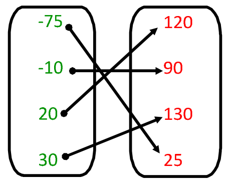
A RELATIONS AND FUNCTIONS

Relation: Pairs of items that are related in a predictable way.

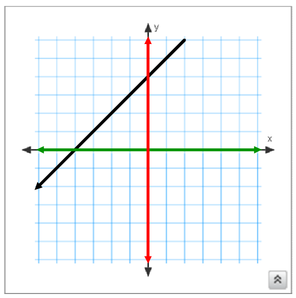
Example: You have $100 in a bank account. You make a deposit or withdrawal. There is a ***relation*** between the balance in the bank and the transaction.

Relations have an "**input**" and an "**output**"

Input (Transaction)	Output (Balance)
20	120
-10	90
30	130
-75	25



(20, 120), (-10, 90), (30, 130), (-75, 25)



The set of all possible "inputs" is called the "**Domain**" of the relation.

The set of all possible "outputs" is called the "**Range**" of the relation.

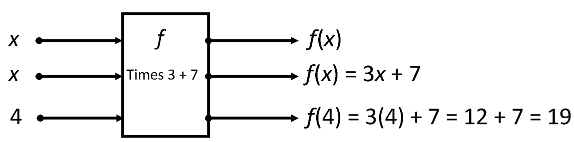
$y = 100 + x$

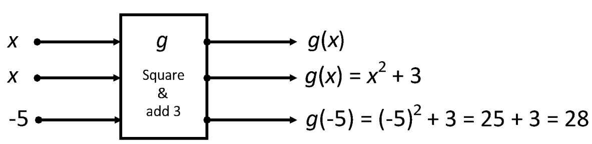
$f(x) = 100 + x$

$f : x \mapsto 2x + 5$

Notation: If $f(x) = 100 + x$, then $f(2) = 100 + 2 = 102$

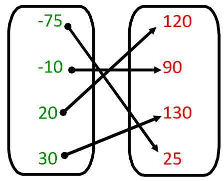
Think of a "function" machine that performs some operation:



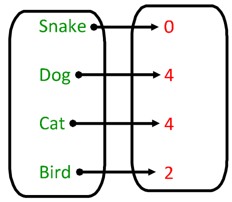


The output value is sometimes called the ***image*** of *x*.

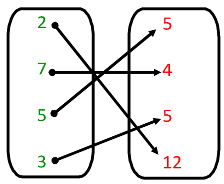
A ***function*** is a relation that has exactly one output for every input.

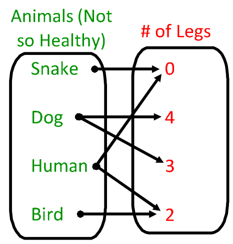


Is this a *function*?



Is this a *function*?



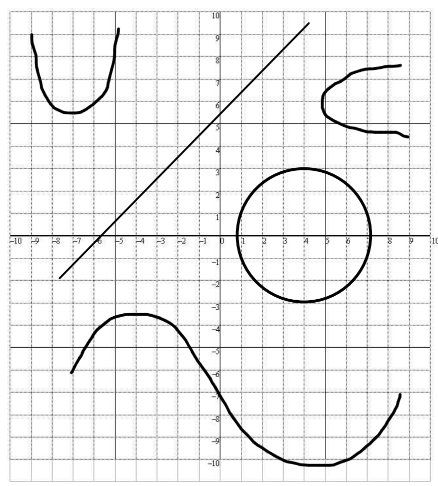


Are these *functions*?

(5, 6), (4, 5), (3, 6), (2, 5)

(1, 6), (2, 5), (1, 4), (5, 5)

Determining whether a relation is a function is easy from a graph:

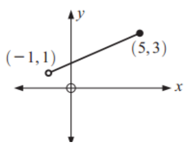


Vertical Line Test: If a vertical line passes through the graph of a relation exactly once for all *x* in its domain, then the relation is a function in that domain.

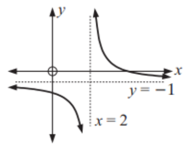
C DOMAIN AND RANGE

Domain: The set of all possible values of the input (*x*) to a relation.
Range: The set of all possible values of the output (*y*) of a relation.

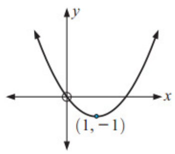
Notation:
Open circle means ***exclude*** the point.
Solid circle means ***include*** the point.
$\{x \mid x \in \mathbb{R}\}$ $\{x \mid x > 2\}$ $\{x \mid 0 \le x \le 12\}$ $\{y \mid y < -6 \text{ or } y \ge 6\}$ $\{y \mid y \in \{2,3,4\}\}$



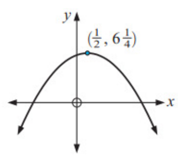
$Domain: \{x \mid -1 < x \le 5\}$
$Range: \{y \mid 1 < y \le 3\}$



$Domain: \{x \mid x \ne 2\}$
$Range: \{y \mid y \ne -1\}$



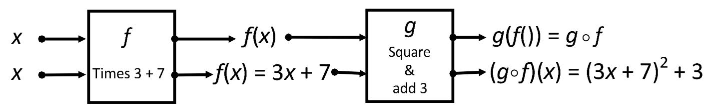
$Domain: \{x \mid x \in \mathbb{R}\}$
$Range: \{y \mid y \ge -1\}$



$Domain: \{x \mid x \in \mathbb{R}\}$
$Range: \{y \mid y \le 6\frac{1}{4}\}$

1A: 1-4
1B: 2ace,5all,6,9,11
1C: 1adghi,2,3all,4jkl

D COMPOSITE FUNCTIONS



The composite of two functions is created by using the output of one function as the input to the other function. Some properties:

$(f \circ g)(x)$ is ***not*** the same as $(g \circ f)(x)$ in general

The range of the first function in a composition is the domain of the second.

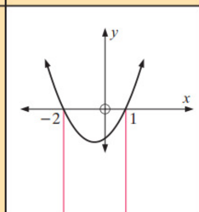
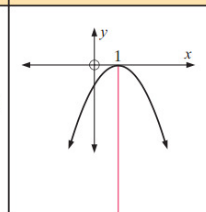
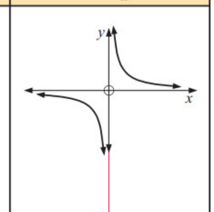
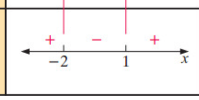
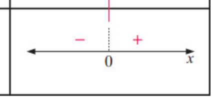
Try: Given $f(x) = x^2 + 7$ and $g(x) = 2x + 4$ find $(f \circ g)(x)$ and $(g \circ f)(x)$

$(f \circ g)(x) = (2x + 4)^2 + 7$ $\qquad$ $(g \circ f)(x) = 2(x^2 + 7) + 4$

1D: 1-4

E SIGN DIAGRAMS

Sometimes it's helpful to visualize a function. Short of creating a full graph of it, one can create a ***sign diagram*** showing just the sign of the function over its domain.

Function	$y=(x+2)(x-1)$	$y=-2(x-1)^2$	$y=\frac{4}{x}$
Graph			
Sign diagram			

The horizontal line represents the *x*-axis

The ***critical values*** are written below the horizontal line. They are the *x*-intercepts and vertical asymptotes of the function.

The sign of the function (+ or -) is written between critical values above the horizontal line.

Notice that the sign ***does not always change between critical values***.

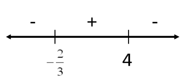
Factoring (writing as a product) is a very helpful tool in creating sign diagrams because if any factor is zero, then the whole expression is zero!

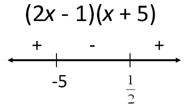
Notice how vertical ***asymptotes*** are represented. A vertical asymptote will occur ***at any value of x that causes the function to divide by zero.***

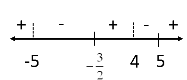
Draw a sign diagram for:

$(3x+2)(4-x)$ $\quad 2x^2+9x-5$ $\quad \dfrac{(2x+3)(x-5)}{(x-4)(x+5)}$

$(2x-1)(x+5)$



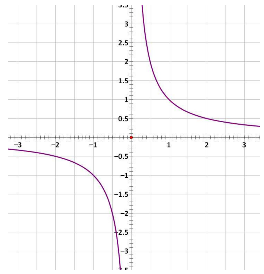




1E: 1cfil,2cfilo,3cfi,4cfil

F THE RECIPROCAL FUNCTION

The function $f(x)=\frac{1}{x}$ is called the ***Reciprocal Function***.



G ASYMPTOTES OF OTHER RATIONAL FUNCTIONS

It generates a whole family of functions that are known as ***Rational Functions*** because they are written as ratios.

They often have ***vertical asymptotes***:
at values of *x* where the denominator is zero.

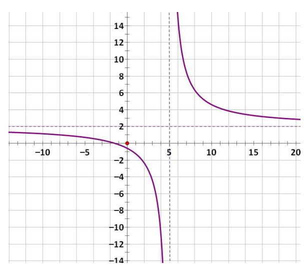
They may also have ***zeros or roots***:
at values of *x* where the numerator is zero.

They may also have ***horizontal asymptotes***:
a *y* value that the function approaches as *x* gets very large or small.

Consider the function $\frac{(2x+3)}{(x-5)}$

Vertical asymptote at ?

Horizontal asymptote at ?



Notation:
Horizontal asymptotes
as $x \to \infty$, $y \to 2$ from above or $y \to 2^+$
as $x \to -\infty$, $y \to 2$ from below or $y \to 2^-$

Vertical asymptotes
as $x \to 5$ from the left, $y \to \infty$ or as $x \to 5^-$, $y \to \infty$
as $x \to 5$ from the right, $y \to \infty$ or as $x \to 5^+$, $y \to \infty$

1F: 1-2
1G: 1all

H INVERSE FUNCTIONS

What is the ***inverse*** of

addition?	multiplication?	exponentiation?
subtraction?	division?	taking roots?

Inverses ***undo*** an operation.

An ***inverse function*** ***undoes*** the operation of a forward function.

Example:
f(*x*) = 3*x* + 2 multiplies *x* by 3, then adds two

The inverse function, called $f^{-1}(x)$ has to subtract 2 then divide by 3

so $f^{-1}(x) = \frac{x-2}{3}$

Some properties of inverse functions:

The inverse of *f*(*x*) must itself be a function (satisfies vertical line test)

The inverse of *f*(*x*) is the reflection of *f*(*x*) over the line *y* = *x*!

$f^{-1} \circ f = f \circ f^{-1} = x$

The domain of f^{-1} is the range of *f*
The domain of *f* is the range of f^{-1}

A ***self-inverse function*** is it's own inverse!
Any function that is symmetric around *y* = *x* is a self-inverse.

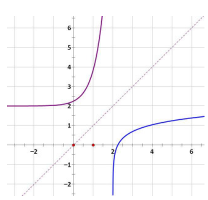
Finding inverse functions

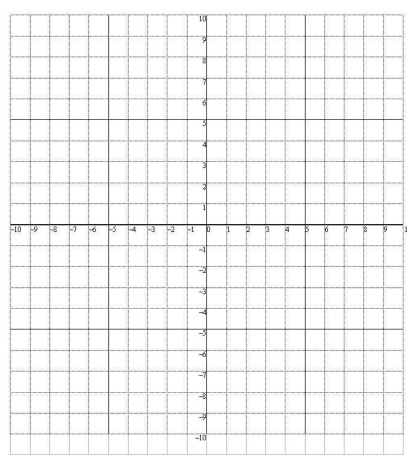
Graphically
Graph the reflection over *y* = *x*

Algebraically
Switch the *x* and *y*, then solve for *y*.

Example:

$y = \frac{1}{4}e^{2x} + 2$





$f(x) = y = \frac{1}{4}e^{2x} + 2$ *so switch x and y to begin:*

$x = \frac{1}{4}e^{2y} + 2$

$x - 2 = \frac{1}{4}e^{2y}$

$4(x-2) = e^{2y}$

$\ln(4(x-2)) = 2y$

$\frac{1}{2}\ln(4(x-2)) = y$

or $f^{-1}(x) = \frac{1}{2}\ln(4(x-2))$

1H: 2,4,5,6,7,8,10,11,12,14