Chapter 10

Advanced trigonometry

A OBSERVING PERIODIC BEHAVIOUR

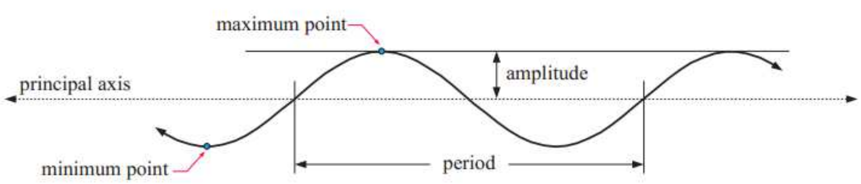
Trig functions are examples of periodic functions because they repeat. All periodic functions have certain common characteristics.

A **periodic function** is one which repeats itself over and over in a horizontal direction.

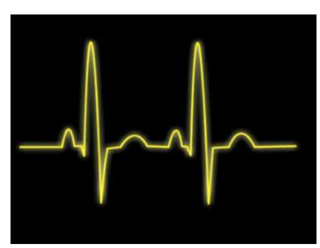
The **period** of a periodic function is the length of one repetition or cycle.

$f(x)$ is a periodic function with period $p \Leftrightarrow \boldsymbol{f(x+p) = f(x)}$ for all x, and p is the smallest positive value for this to be true.

amplitude $= \dfrac{\textbf{max} - \textbf{min}}{2}$ **principal axis** $y = \dfrac{\textbf{max} + \textbf{min}}{2}$



The ***sine wave*** is a common term for a periodic function. But not all periodic functions are sine waves or even trig functions.



B THE SINE FUNCTION

A problem, complements of IMP Year 4

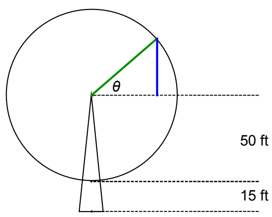
In a circus act, a performer stands on a platform on a Ferris wheel whose center is 65 feet off the ground, has a radius 50 ft and turns counterclockwise at a constant rate making one complete revolution every 40 seconds. At the moment the platform passes the 3:00 position, a cart full of water, located 240 feet to the left of the center of the base of the Ferris wheel starts moving to the right at a speed of 15 feet per second. At just the right moment, the performer steps gently off the platform and falls into the cart of water. How many seconds after the platform passes the 3:00 position should the diver step off the platform?

Beginning on a new page in your notebook, ***record the important information and draw a sketch***, labelling the key values.

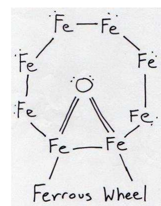
Leave several blank pages in your notebook after this one as we will come back to this problem.

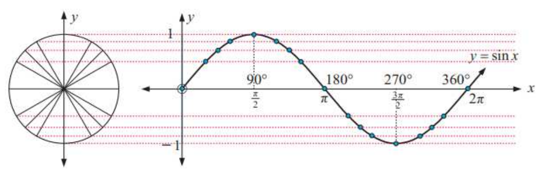
We'll start with the question of "How high off the ground is the diver when the radius to his platform makes an angle of θ with the horizontal?" How does his height change as θ changes? **Write *h* as a function of *θ*.**

GSP Demo   Geogebra Trig Tracers



$h(t) = 65 + 50\sin\theta$





In our example, the angle θ depends on time so we can think of the sine curve as a ***function of time*** rather than as a function of an angle. **Write *θ* as a function of *t*.**

$$\text{Angular rate} = \frac{2\pi\ \text{rad}}{40\ \text{sec}} = \frac{\pi\ \text{rad}}{20\ \text{sec}} \quad \text{so } \theta = \text{rate} \bullet \text{time} = \frac{\pi}{20} \bullet t$$

Now ***write h as a function of t***.

$$h(t) = 65 + 50\sin\left(\frac{\pi}{20} \bullet t\right)$$

How would this function change if the Ferris Wheel were larger or smaller?

The coefficient multiplying the sin function controls the ***amplitude*** or height of the function.

How would this function change if the Ferris Wheel were spinning faster?

The coefficient multiplying the ***variable inside*** the sin function controls the ***period*** of the function.

More generally, we can think of the ***sine function*** as a function of any input variable, say *x*.

- in $y = a\sin x$, $|a|$ determines the amplitude
- in $y = \sin bx$, $b > 0$, b affects the period and the period is $\frac{2\pi}{b}$.

10A: #2,3 (Periodic behavior)
10B.1: #1,3,4,6 (The sine curve)

Transformations of the Sine function   Geogebra

We have seen that in the function $y = a\sin(bx)$ the parameters a and b control the ***amplitude*** and ***period*** of the sine curve.

Notice that a and b ***multiply*** the function and its argument respectively.
That is, if $f(x) = \sin(x)$, we explored $g(x) = a \cdot f(bx)$.

Recall from our work with transformations that:

> A value ***multiplied*** by ***a function*** represents a ***vertical scale*** (stretch or compress)
>
> A value ***multiplied*** by ***the argument of a function*** represents a ***horizontal scale.***

Back to our ferris wheel. The function was: $h(t) = 65 + 50\sin(\frac{\pi}{20} \cdot t)$

What does the 65 represent? It is the value of the principle axis.

What would happen if we increased or decreased the height of the center of the ferris wheel?

> A value ***added*** to ***a function*** represents a ***vertical shift*** or ***offset***.

Finally, consider what would happen if the ferris wheel was at the top at $t = 0$.
At $t = 0$, we want to use $\sin(\frac{\pi}{2})$. Notice that we need to ***add*** 10 seconds to all the times.

$$h(t) = 65 + 50\sin(\frac{\pi}{20} \cdot (t + 10))$$

Again, recall from our work with transformations:

> A value ***added*** to ***the argument of a function*** represents a ***horizontal or phase shift***.

GSP Demo

The General Sine Function

$y = a \sin b(x - c) + d$ is called the **general sine function**.

a	b	c	d
affects **amplitude**	affects **period**	affects **horizontal translation**	affects **vertical translation**

The **principal axis** of the general sine function is $y = d$.

The **period** of the general sine function is $\frac{2\pi}{b}$.

Note the subtle difference between $a\sin b(x - c)$ and $a\sin(bx - c)$
Different books & problems do it differently. Need to watch out. In the first case, the sine wave starts (is at the beginning of a new cycle) at $x = c$. In the other, it starts at $x = c/b$. You generally know when the cycle is supposed to start so put the parentheses in the place that makes it easiest to understand.

C MODELLING USING SINE FUNCTIONS

Some periodic phenomena can be modeled with a sine curve. Examples:
- The height of a tide over several days
- The number of hours of daylight over a year
- The average temperature over a year

To use a sine curve as a model, you need to find four parameters:
- The ***amplitude*** of the phenomenon
- The ***offset*** of the phenomenon
- The ***period*** of the phenomenon = duration of one complete cycle
- The ***phase*** of the phenomenon

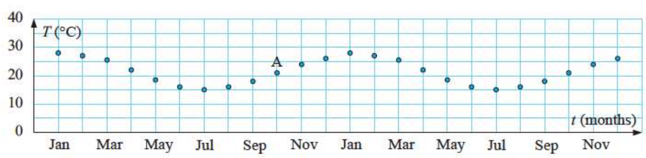
Then construct the function:
$y = a\sin(b(t - c) + d$ with

- a = the ***amplitude*** of the phenomenon = (max - min)/2
- d = the ***offset*** of the phenomenon = (min + max)/2
- $b = 2\pi/$***period***
- c = a value such that when $t = 0$ the value of the function is correct

Consider again the mean monthly maximum temperature for Cape Town:

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temp (°C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

The graph over a two year period is shown below:



$T = a\sin b(t - c) + d.$

The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore\ b = \frac{\pi}{6}$.

The amplitude $= \frac{\max - \min}{2} \approx \frac{28 - 15}{2} \approx 6.5$, so $a \approx 6.5$.

The principal axis is $d \approx \frac{28 + 15}{2} \approx 21.5$

$T \approx 6.5\sin\frac{\pi}{6}(t - c) + 21.5$ for some constant c.

$t = 10$ is the start of a new cycle so...

$T \approx 6.5\sin\frac{\pi}{6}(t - 10) + 21.5$

Summary

To find a trig function that fits cyclical data, you must identify the four parameters:
- Amplitude = vertical stretch
- Frequency (or period) = horizontal stretch
- Phase = horizontal shift
- Offset (principal axis) = vertical shift

10B.2: #1def,2,3,4bdfhj (Xforms of sine)
10C: #1-5 (Modeling with sine)
QB #63,78 (a-c), 8 (a-d)

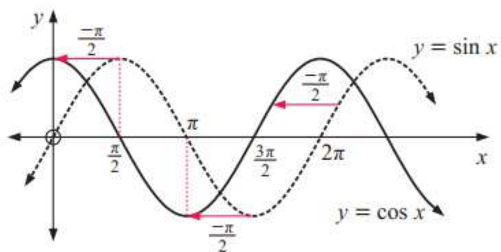
Present #1-5 all

D THE COSINE FUNCTION

Let's look at the cosine function next:

$$\cos x = \sin\left(x + \tfrac{\pi}{2}\right).$$

$$-\cos x = \sin\left(x - \tfrac{\pi}{2}\right).$$



One can describe all the same transformations for the cosine function. However, since cosine is just a phase shift of sine, we generally use the sine function in our models.

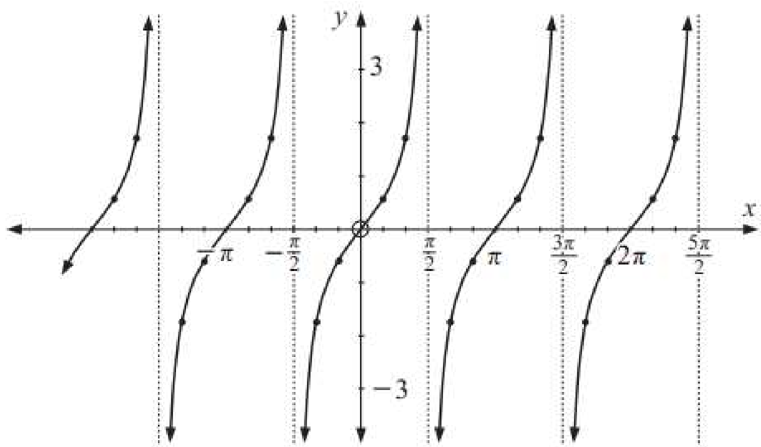
10D: #1ghijkl,2-4 (Graphs of cosine)

Quiz: Graph something like the following:
1. $2\cos(x - \pi/4) + 3$
2. $\sin(3x)$

E THE TANGENT FUNCTION

Let's look at the tangent function next:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$



We observe that $y = \tan x$ has a **period** of π, **range** $y \in \mathbb{R}$, and **vertical asymptotes** $x = \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$.

Can you predict what $\tan(3x)$ would look like?
- Where would the asymptotes be?

10E.1: #1dh,2 (Simple tangent)
10E.2: #1,2(predict, then graph),3 (Tangent transformations)
QB: #6,11,18,21,25,28 (Graphs)

Mr. Alei's Math Page

Algebra 2 Pre DP SL 1 SL 2 HL 2 Resources

Standard Level Year 1

Mostly working on explorations this week. It's due ***Friday, 5/3***. Be sure to talk to me early if you are getting bogged down. We'll pick up with trig again at the end of the week, but mostly just summarizing what we already know. See you in class.
Mr. Alei

SL 1 Assignments: Week of 4/29/13 (Last week - 4/22/13)

Mon 4/29: Explorations in class or in red lab
Tue 4/30: Explorations in class or in blue lab
Thu 5/2: Explorations in class or in blue lab
10F: #1-2,3-6 3rd col, 7,8 (General Trig functions)

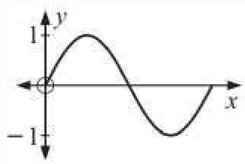
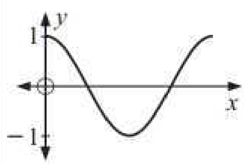
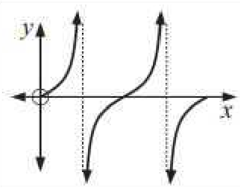
Note:
Fri 5/3: *Deadline for turning in exploration for full DA credit*

Trig Unit Plan
Triangle Trig: Smartboard Notes
Trig Functions: Smartboard Notes
QB Trig Practice MarkScheme

Exploration Timeline
Draft Due: Thu 4/11
Final Due: Tue 4/30
Exploration Resources
Oxford chapter on Exploration
Example papers (score shown in parens)
Ex 1: Codes (15) Comments
Ex 2: Totient Theorem (16) Comments
Ex 4: Minesweeper (5) Comments
Ex 4: Music (9) Comments
Ex 5: Newton Raphson (11) Comments
Ex 6: Polar Diagrams (20) Comments
Ex 7: Rainfall (16) Comments
Ex 8: Spirals (16) Comments
Ex 9: Tower of Hanoi (14) Comments
Topic Ideas
sport, archaeology, computers, algorithms, cell phones, music, sine, musical harmony

F GENERAL TRIGONOMETRIC FUNCTIONS

FEATURES OF CIRCULAR FUNCTIONS

Function	*Sketch for* $0 \leqslant x \leqslant 2\pi$	*Period*	*Amplitude*	*Domain*	*Range*
$y = \sin x$		2π	1	$x \in \mathbb{R}$	$-1 \leqslant y \leqslant 1$
$y = \cos x$		2π	1	$x \in \mathbb{R}$	$-1 \leqslant y \leqslant 1$
$y = \tan x$		π	undefined	$x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$	$y \in \mathbb{R}$

GENERAL TRIGONOMETRIC FUNCTIONS

General function	a affects vertical stretch	$b > 0$ affects horizontal stretch	c affects horizontal translation	d affects vertical translation
$y = a\sin b(x-c) + d$ $y = a\cos b(x-c) + d$	amplitude $= \lvert a \rvert$	period $= \frac{2\pi}{b}$	• $c > 0$ moves the graph right • $c < 0$ moves the graph left	• $d > 0$ moves the graph up • $d < 0$ moves the graph down
$y = a\tan b(x-c) + d$	amplitude undefined	period $= \frac{\pi}{b}$		• principal axis is $y = d$

10F: #1-2,3-6 3rd col, 7,8 (General Trig functions)

Verbal go-round on 10F

G TRIGONOMETRIC EQUATIONS

It can get triggy.....so pay attention!

Trig functions often arise in equations. A simple example is:

$$\cos(\theta) = \tfrac{1}{2}$$

The obvious solution is $\theta = \frac{\pi}{3}$. Try your calculator. It will say $\frac{\pi}{3}$ (or 60°)

But we know that there are other solutions! For example, $\theta = -\frac{\pi}{3}, \frac{7\pi}{3}$, etc.

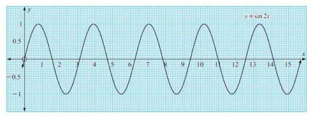
> In general trig equations can have an infinite number of solutions unless the domain is specified. If not specified, you must represent all solutions!

We write the general solution as $\pm\frac{\pi}{3} + 2\pi k$ with $k \in \mathbf{Z}$

> ***Unless otherwise specified, trig equations are assumed to use radians for the variables!***

More complex trig equations: Graphical solutions
Sometimes you can solve a trig equation by reading from a graph.

- Read the values from the graph.
- Use a ruler to draw clean lines to the axes.



Use the graph of $y = \sin 2x$ to find, correct to 1 decimal place, the solutions of:
a $\sin 2x = 0.7,\ 0 \leqslant x \leqslant 16$ **b** $\sin 2x = -0.3,\ 0 \leqslant x \leqslant 16.$

Solutions using technology
One way to solve trig equations is to use a graphing calculator.

- Graph the expressions on each side of the "=" sign.
- Use CALC/INTERSECT to find the intersections which are the solutions
- Don't forget to restrict yourself to the domain given or describe the complete solution set.

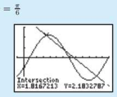
Solve $2\sin x - \cos x = 4 - x$ for $0 \leqslant x \leqslant 2\pi$.

We graph the functions $Y_1 = 2\sin X - \cos X$ and $Y_2 = 4 - X$ on the same set of axes.

We need to use **window** settings just larger than the domain.

In this case, $X\text{min} = -\frac{\pi}{6}$ $X\text{max} = \frac{13\pi}{6}$ $X\text{scale} = \frac{\pi}{6}$

The grid facility on the graphics calculator can also be helpful, particularly when a sketch is required.



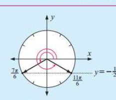
Using the appropriate function on the calculator gives the following solutions:

$$x \approx 1.82,\ 3.28,\ 5.81$$

Solving trig equations analytically
- When an exact answer is required!
- They will generally involve known angles. Give exact surd (√)answers when possible.
- Remember periodicity - there are usually multiple answers. Read the question!

Use the unit circle to find the exact solutions of x, $0 \leqslant x \leqslant 3\pi$ for:
a $\sin x = -\frac{1}{2}$ **b** $\sin 2x = -\frac{1}{2}$ **c** $\sin\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$

a $\sin x = -\frac{1}{2}$, so from the unit circle



$$x = \left.\begin{matrix}\frac{7\pi}{6}\\ \frac{11\pi}{6}\end{matrix}\right\} + k2\pi, \quad k \text{ an integer}$$

$$\therefore\quad x = \underset{k=0}{\frac{7\pi}{6}},\ \underset{k=0}{\frac{11\pi}{6}},\ \underset{k=1}{\cancel{\frac{19\pi}{6}}} \text{ is too big}$$

Substituting $k = 1, 2, 3, ...$ gives answers outside the required domain.
Likewise $k = -1, -2, ...$ gives answers outside the required domain.
$\therefore$ there are *two solutions*: $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

b $\sin 2x = -\frac{1}{2}$ is solved in exactly the same way:

$$2x = \left.\begin{matrix}\frac{7\pi}{6}\\ \frac{11\pi}{6}\end{matrix}\right\} + k2\pi, \quad k \text{ an integer}$$

$$\therefore\quad x = \left.\begin{matrix}\frac{7\pi}{12}\\ \frac{11\pi}{12}\end{matrix}\right\} + k\pi \qquad \{\text{dividing each term by } 2\}$$

$$\therefore\quad x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{31\pi}{12}, \frac{35\pi}{12} \qquad \{\text{obtained by letting } k = 0, 1, 2\}$$

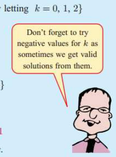
c $\sin(x - \frac{\pi}{6}) = -\frac{1}{2}$ is also solved in the same way:

$$x - \frac{\pi}{6} = \left.\begin{matrix}\frac{7\pi}{6}\\ \frac{11\pi}{6}\end{matrix}\right\} + k2\pi$$

$$\therefore\quad x = \left.\begin{matrix}\frac{8\pi}{6}\\ 2\pi\end{matrix}\right\} + k2\pi \qquad \{\text{adding } \tfrac{\pi}{6} \text{ to both sides}\}$$

$$\therefore\quad x = \underset{k=0}{\frac{4\pi}{3}},\ \underset{k=0}{2\pi},\ \underset{k=1}{\cancel{\frac{10\pi}{3}}} \text{ is too big},\quad \underset{k=-1}{0}$$

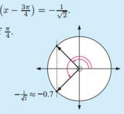
So, $x = 0$, $\frac{4\pi}{3}$ or 2π which is *three solutions*.



Find exact solutions of $\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) + 1 = 0$ for $0 \leqslant x \leqslant 6\pi$.

Rearranging $\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) + 1 = 0$, we find $\cos\left(x - \frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

We recognise $\frac{1}{\sqrt{2}}$ as a special fraction for multiples of $\frac{\pi}{4}$.



$$\therefore\quad x - \frac{3\pi}{4} = \left.\begin{matrix}\frac{3\pi}{4}\\ \frac{5\pi}{4}\end{matrix}\right\} + k2\pi, \quad k \text{ an integer}$$

$$\therefore\quad x = \left.\begin{matrix}\frac{3\pi}{2}\\ 2\pi\end{matrix}\right\} + k2\pi$$

If $k = -1$, $x = -\frac{\pi}{2}$ or 0. If $k = 0$, $x = \frac{3\pi}{2}$ or 2π.
If $k = 1$, $x = \frac{7\pi}{2}$ or 4π. If $k = 2$, $x = \frac{11\pi}{2}$ or 6π.

If $k = 3$, the answers are greater than 6π.

So, the solutions are: $x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, 4\pi, \frac{11\pi}{2}$ or 6π.

> 10G.1: #1-4 (Trig equations from graphs)
> 10G.2: #1-2 (Trig equations w/calculator)
> 10G.3: #1,2bdfhj,4-6 (Trig equations analytically)
> QB: #31,34,35,40,47,48,59 (Graphs)

H USING TRIGONOMETRIC MODELS

No real new ideas here, we'll explore how we can use trig functions to model real world ideas.

The height $h(t)$ metres of the tide above mean sea level on January 24th at Cape Town is modelled approximately by $h(t) = 3\sin\left(\frac{\pi t}{6}\right)$ where t is the number of hours after midnight.

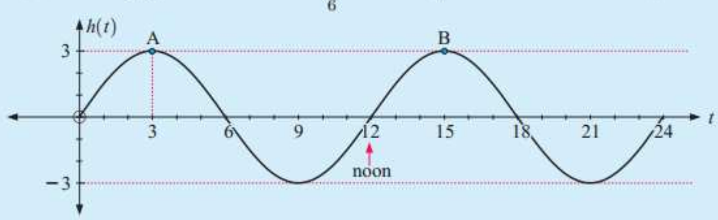
a Graph $y = h(t)$ for $0 \leqslant t \leqslant 24$.

b When is high tide and what is the maximum height?

c What is the height at 2 pm?

d If a ship can cross the harbour provided the tide is at least 2 m above mean sea level, when is crossing possible on January 24?

a $h(t) = 3\sin\left(\frac{\pi t}{6}\right)$ has period $= \dfrac{2\pi}{\frac{\pi}{6}} = 2\pi \times \frac{6}{\pi} = 12$ hours and $h(0) = 0$

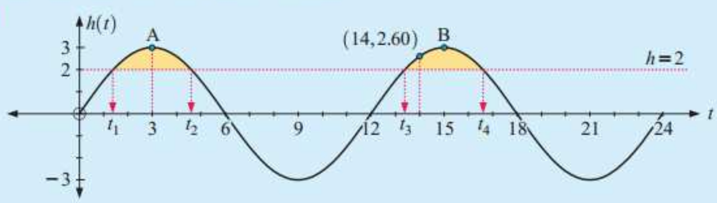


b High tide is at 3 am and 3 pm. The maximum height is 3 m above the mean as seen at points A and B.

c At 2 pm, $t = 14$ and $h(14) = 3\sin\left(\frac{14\pi}{6}\right) \approx 2.60$ m

So, the tide is 2.6 m above the mean.

d



We need to solve $h(t) = 2$, so $3\sin\left(\frac{\pi t}{6}\right) = 2$.

Using a graphics calculator with $Y_1 = 3\sin\left(\frac{\pi X}{6}\right)$ and $Y_2 = 2$

we obtain $t_1 = 1.39$, $t_2 = 4.61$, $t_3 = 13.39$, $t_4 = 16.61$

or you could **trace** across the graph to find these values.

Now 1.39 hours = 1 hour 23 minutes, and so on.

So, the ship can cross between 1:23 am and 4:37 am or 1:23 pm and 4:37 pm.

I TRIGONOMETRIC RELATIONSHIPS

When working with trig functions in equations, you need to know how to manipulate them. Since sinθ, cosθ, etc. are real numbers, you treat them just as you would a variable.

For example:

sinθ + sinθ = 2sinθ
5sinθ - 12sinθ = -7sinθ
5sinθ - $x$sinθ = (5 - x)sinθ
(5sinθ)(4sinθ) = 20sin$^2\theta$ etc.

But notice that:

sinθ + sin2θ ≠ sin3θ we don't add or multiply the ***argument*** of the functions.
sinθ + cosθ cannot be simplified further
sin(θ + t) ≠ sinθ + sin(t) the "function" does not "distribute".
sin$x$ means sin ***of*** x, not sin times x (think about it - what is sin without an argument?

Simplify: **a** $3\cos\theta + 4\cos\theta$ **b** $\sin\alpha - 3\sin\alpha$

a Since $3x + 4x = 7x$,
$3\cos\theta + 4\cos\theta = 7\cos\theta$

b Since $x - 3x = -2x$,
$\sin\alpha - 3\sin\alpha = -2\sin\alpha$

A relationship that is always true is called an ***identity***. There are many trig identities that allow you to simplify, rearrange, and ultimately solve equations. You need to learn the main ones and make use of them. Knowing which to use when takes practice and experience. (aka doing HW with an eye to the meaning and connections). Perhaps the most fundamental and useful trig identity is

$$\sin^2\theta + \cos^2\theta = 1$$

The big Kahuna! aka ***The Pythagorean Identity***

This can be rearranged in various ways as needed.

$\sin^2\theta = 1 - \cos^2\theta$ $1 - \cos^2\theta = \sin^2\theta$
$\cos^2\theta = 1 - \sin^2\theta$ $1 - \sin^2\theta = \cos^2\theta$

Simplify: **a** $2 - 2\sin^2\theta$ **b** $\cos^2\theta\sin\theta + \sin^3\theta$

a
$$\begin{aligned} &2 - 2\sin^2\theta \\ &= 2(1 - \sin^2\theta) \\ &= 2\cos^2\theta \\ &\{\text{as } \cos^2\theta + \sin^2\theta = 1\} \end{aligned}$$

b
$$\begin{aligned} &\cos^2\theta\sin\theta + \sin^3\theta \\ &= \sin\theta(\cos^2\theta + \sin^2\theta) \\ &= \sin\theta \times 1 \\ &= \sin\theta \end{aligned}$$

Expand and simplify: $(\cos\theta - \sin\theta)^2$

$$\begin{aligned} &(\cos\theta - \sin\theta)^2 \\ &= \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta \quad \{\text{using } (a-b)^2 = a^2 - 2ab + b^2\} \\ &= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta \\ &= 1 - 2\cos\theta\sin\theta \end{aligned}$$

10H: #1-5 (Trig models)
10I.1: #1-4 (Simplify sums)
QB: #62,64,66,70,71,72 (Graphs)

I TRIGONOMETRIC RELATIONSHIPS

Continued - 10.I.2 - more complex manipulations

Things get more interesting, of course. In many cases our old friend ***factoring*** can help us.

Factorise: **a** $\cos^2\alpha - \sin^2\alpha$ **b** $\tan^2\theta - 3\tan\theta + 2$

a $\cos^2\alpha - \sin^2\alpha$

$= (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha)$ $\{a^2 - b^2 = (a+b)(a-b)\}$

b $\tan^2\theta - 3\tan\theta + 2$

$= (\tan\theta - 2)(\tan\theta - 1)$ $\{x^2 - 3x + 2 = (x-2)(x-1)\}$

Big deal. Why would we ever want to factor trig expressions? Well, besides the sheer fun of it, sometimes it enables you to simplify things:

Simplify: **a** $\dfrac{2 - 2\cos^2\theta}{1 + \cos\theta}$ **b** $\dfrac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}$

a
$$\frac{2 - 2\cos^2\theta}{1 + \cos\theta}$$
$$= \frac{2(1 - \cos^2\theta)}{1 + \cos\theta}$$
$$= \frac{2\cancel{(1 + \cos\theta)}(1 - \cos\theta)}{\cancel{(1 + \cos\theta)}}$$
$$= 2(1 - \cos\theta)$$

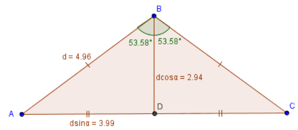
b
$$\frac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}$$
$$= \frac{\cancel{(\cos\theta - \sin\theta)}}{(\cos\theta + \sin\theta)\cancel{(\cos\theta - \sin\theta)}}$$
$$= \frac{1}{\cos\theta + \sin\theta}$$

The value of this will become more apparent as we learn more trig identities.

10I.2: #1-3 (Factor trig)
QB: #16,36 (Equations)

J DOUBLE ANGLE FORMULAE

Let's look at trig functions of double angles. Consider an isosceles triangle:



$$TotalArea = 2 \cdot \frac{1}{2}hb$$
$$= dsin\alpha \cdot dcos\alpha = d^2 sin\alpha \cdot cos\alpha$$
$$= 4.96 * 3.99 * 2.94 = 11.75$$

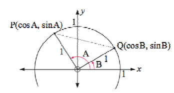
$$TotalArea = \frac{1}{2}absin\theta$$
$$= \frac{1}{2}d^2 sin(2\alpha)$$
$$= .5 * 24.6 * 0.96 = 11.75$$

Since d^2 sinα cosα = ½ d^2 sin(2α) we can cancel the d^2 and multiply both sides by two to get:

sin(2α) = 2sinα cosα

What about cosine? Not so straightforward. But we can get there from looking at the sum or difference of two angles.

Consider P$(\cos A, \sin A)$ and Q$(\cos B, \sin B)$ as any two points on the unit circle, as shown.



Angle POQ is $A - B$.

Using the distance formula:

$$PQ = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$

$$\therefore \quad (PQ)^2 = \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$$
$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$$
$$= 2 - 2(\cos A \cos B + \sin A \sin B) \quad ...... (1)$$

But, by the *cosine rule* in ΔPOQ,

$$(PQ)^2 = 1^2 + 1^2 - 2(1)(1)\cos(A - B)$$
$$= 2 - 2\cos(A - B) \quad ...... (2)$$

$$\therefore \quad \cos(A - B) = \cos A \cos B + \sin A \sin B \qquad \{\text{comparing (1) and (2)}\}$$

From this formula the other formulae can be established:

$$\cos(A + B) = \cos(A - (-B))$$
$$= \cos A \cos(-B) + \sin A \sin(-B)$$
$$= \cos A \cos B + \sin A(-\sin B) \qquad \{\cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta\}$$
$$= \cos A \cos B - \sin A \sin B$$

This formula, and the corresponding one for sin(A + B) are exceptionally useful. But for some reason, they are not included in the Math SL syllabus. However, we can use it to find a formula that ***is*** in the SL formula, the double angle formula for cosine. To do this let the two angles A and B be the same in the above:

cos(A + A) = cos(2A) = cos$A$cos$A$ - sin$A$sin$A$ = cos^2A - sin^2A

Making use of the Pythagorean Identity, we have the following useful forms:

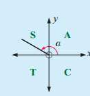
Double Angle Formulae

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$= 2\cos^2\theta - 1$$

Let's see how we can use these.

If $\sin\alpha = \frac{5}{13}$ where $\frac{\pi}{2} < \alpha < \pi$, find the value of $\sin 2\alpha$ without using a calculator.

α is in quadrant 2, so $\cos\alpha$ is negative.



Now $\cos^2\alpha + \sin^2\alpha = 1$

$\therefore \quad \cos^2\alpha + \frac{25}{169} = 1$

$\therefore \quad \cos^2\alpha = \frac{144}{169}$

$\therefore \quad \cos\alpha = \pm\frac{12}{13}$

$\therefore \quad \cos\alpha = -\frac{12}{13}$

But $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2(\frac{5}{13})(-\frac{12}{13})$

$= -\frac{120}{169}$

Given that $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = -\frac{4}{5}$ find:

a $\sin 2\alpha$ **b** $\cos 2\alpha$

a $\sin 2\alpha$
$= 2\sin\alpha\cos\alpha$
$= 2(\frac{3}{5})(-\frac{4}{5})$
$= -\frac{24}{25}$

b $\cos 2\alpha$
$= \cos^2\alpha - \sin^2\alpha$
$= (-\frac{4}{5})^2 - (\frac{3}{5})^2$
$= \frac{7}{25}$

If α is acute and $\cos 2\alpha = \frac{3}{4}$ find the values of: **a** $\cos\alpha$ **b** $\sin\alpha$.

a $\cos 2\alpha = 2\cos^2\alpha - 1$

$\therefore \quad \frac{3}{4} = 2\cos^2\alpha - 1$

$\therefore \quad \cos^2\alpha = \frac{7}{8}$

$\therefore \quad \cos\alpha = \pm\frac{\sqrt{7}}{2\sqrt{2}}$

$\therefore \quad \cos\alpha = \frac{\sqrt{7}}{2\sqrt{2}}$

$\{\text{as } \alpha \text{ is acute, } \cos\alpha > 0\}$

b $\sin\alpha = \sqrt{1 - \cos^2\alpha}$

$\{\text{as } \alpha \text{ is acute, } \sin\alpha > 0\}$

$\therefore \quad \sin\alpha = \sqrt{1 - \frac{7}{8}}$

$\therefore \quad \sin\alpha = \sqrt{\frac{1}{8}}$

$\therefore \quad \sin\alpha = \frac{1}{2\sqrt{2}}$

Use an appropriate 'double angle formula' to simplify:

a $3\sin\theta\cos\theta$ **b** $4\cos^2 2B - 2$

a $3\sin\theta\cos\theta$
$= \frac{3}{2}(2\sin\theta\cos\theta)$
$= \frac{3}{2}\sin 2\theta$

b $4\cos^2 2B - 2$
$= 2(2\cos^2 2B - 1)$
$= 2\cos 2(2B)$
$= 2\cos 4B$

10J: #1-9 (Double angle formulae)
QB: #26,51 (Double angle)

K TRIGONOMETRIC EQUATIONS IN QUADRATIC FORM

Sometimes we encounter trig functions in equations that are in a ***quadratic form***. Consider, for example:

$$2\sin^2 x + \sin x = 0 \qquad \text{and} \qquad 2\cos^2 x + \cos x + 1 = 0$$
$$\therefore \quad \sin x(2\sin x + 1) = 0 \qquad \qquad \therefore \quad (2\cos x - 1)(\cos x + 1) = 0$$
$$\therefore \quad \sin x = 0 \text{ or } -\tfrac{1}{2} \qquad \qquad \therefore \quad \cos x = \tfrac{1}{2} \text{ or } -1$$

We may also need to find the value(s) of x!

$x = 2k\pi, -\pi/6 + 2k\pi, -5\pi/6 + 2k\pi$ for $k \in \mathbb{Z}$

$x = \pi/3 + 2k\pi, -\pi/3 + 2k\pi, \pi + 2k\pi$ for $k \in \mathbb{Z}$

When solving equations involving trig functions look for:
- "Algebra" steps (treat the function like a variable) and
- "Trig" steps (use trig identities and characteristics to rewrite or combine)

Tips for simplifying:
- Treat the *entire function* as a variable when manipulating equations:

  Example : a) $\cos^2\alpha - \sin^2\alpha$ b) $\tan^2\alpha - 3\tan\alpha + 2$

  $(\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha)$ $(\tan\alpha + 1)(\tan\alpha + 2)$

- Know your tools - various formulas to:
  - > Change from sin to cos (shifts of $\pi/2$)
  - > Change the signs from inside to outside the argument ($\sin(-x)$ and $\cos(-x)$)
  - > Change from $\sin^2 x$ to $\sin(2x)$ and $\sin^2 x$ ***or*** $\cos^2 x$ to $\cos(2x)$
  - > Remove $\sin^2$ and $\cos^2$ (Pythagorean Identity)
- Calculator inverse functions return angles in particular ranges:
  - > $-\pi/2 \le \sin^{-1}x \le \pi/2$
  - > $0 \le \cos^{-1}x \le \pi$
  - > $-\pi/2 \le \tan^{-1}x \le \pi/2$

10K: #1-2 (Trig in quad form)
QB: #7,30,42,52,76 (Quadratic form)