Topic 3—Circular functions and trigonometry 16 hours

The aims of this topic are to explore the circular functions and to solve problems using trigonometry. On examination papers, radian measure should be assumed unless otherwise indicated.

	Content	Further guidance	Links
3.1	The circle: radian measure of angles; length of an arc; area of a sector.	Radian measure may be expressed as exact multiples of π, or decimals.	**Int:** Seki Takakazu calculating π to ten decimal places. **Int:** Hipparchus, Menelaus and Ptolemy. **Int:** Why are there 360 degrees in a complete turn? Links to Babylonian mathematics. **TOK:** Which is a better measure of angle: radian or degree? What are the "best" criteria by which to decide? **TOK:** Euclid's axioms as the building blocks of Euclidean geometry. Link to non-Euclidean geometry.
3.2	Definition of $\cos\theta$ and $\sin\theta$ in terms of the unit circle. Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$. Exact values of trigonometric ratios of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples.	The equation of a straight line through the origin is $y = x\tan\theta$. *Examples*: $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$, $\tan 210° = \frac{\sqrt{3}}{3}$.	**Aim 8:** Who really invented "Pythagoras' theorem"? **Int:** The first work to refer explicitly to the sine as a function of an angle is the Aryabhatiya of Aryabhata (ca. 510). **TOK:** Trigonometry was developed by successive civilizations and cultures. How is mathematical knowledge considered from a sociocultural perspective?
3.3	The Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$. Double angle identities for sine and cosine. Relationship between trigonometric ratios.	Simple geometrical diagrams and/or technology may be used to illustrate the double angle formulae (and other trigonometric identities). *Examples*: Given $\sin\theta$, finding possible values of $\tan\theta$ without finding θ. Given $\cos x = \frac{3}{4}$, and x is acute, find $\sin 2x$ without finding x.	
3.4	The circular functions $\sin x$, $\cos x$ and $\tan x$: their domains and ranges; amplitude, their periodic nature; and their graphs. Composite functions of the form $f(x) = a\sin(b(x+c)) + d$. Transformations. Applications.	*Examples*: $f(x) = \tan\left(x - \frac{\pi}{4}\right)$, $f(x) = 2\cos(3(x-4)) + 1$. *Example*: $y = \sin x$ used to obtain $y = 3\sin 2x$ by a stretch of scale factor 3 in the y-direction and a stretch of scale factor $\frac{1}{2}$ in the x-direction. Link to 2.3, transformation of graphs. Examples include height of tide, motion of a Ferris wheel.	**Appl:** Physics 4.2 (simple harmonic motion).
3.5	Solving trigonometric equations in a finite interval, both graphically and analytically. Equations leading to quadratic equations in $\sin x$, $\cos x$ or $\tan x$. **Not required:** the general solution of trigonometric equations.	*Examples*: $2\sin x = 1$, $0 \le x \le 2\pi$, $2\sin 2x = 3\cos x$, $0° \le x \le 180°$, $2\tan(3(x-4)) = 1$, $-\pi \le x \le 3\pi$. *Examples*: $2\sin^2 x + 5\cos x + 1 = 0$ for $0 \le x < 4\pi$, $2\sin x = \cos 2x$, $-\pi \le x \le \pi$.	
3.6	Solution of triangles. The cosine rule. The sine rule, including the ambiguous case. Area of a triangle, $\frac{1}{2}ab\sin C$. Applications.	Pythagoras' theorem is a special case of the cosine rule. Link with 4.2, scalar product, noting that: $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b} \Rightarrow \lvert\boldsymbol{c}\rvert^2 = \lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{b}\rvert^2 - 2\boldsymbol{a}\cdot\boldsymbol{b}$. Examples include navigation, problems in two and three dimensions, including angles of elevation and depression.	**Aim 8:** Attributing the origin of a mathematical discovery to the wrong mathematician. **Int:** Cosine rule: Al-Kashi and Pythagoras. **TOK:** Non-Euclidean geometry: angle sum on a globe greater than 180°.

Unit plan on website has HW assignments for these chapters.

Chapter 8
The unit circle and radian measure

- **A** Radian measure
- **B** Arc length and sector area
- **C** The unit circle and the basic trigonometric ratios
- **D** The equation of a straight line

Chapter 9
Non-right angled triangle trigonometry

- **A** Areas of triangles
- **B** The cosine rule
- **C** The sine rule
- **D** Using the sine and cosine rules

Chapter 10
Advanced trigonometry

- **A** Observing periodic behaviour
- **B** The sine function
- **C** Modelling using sine functions
- **D** The cosine function
- **E** The tangent function
- **F** General trigonometric functions
- **G** Trigonometric equations
- **H** Using trigonometric models
- **I** Trigonometric relationships
- **J** Double angle formulae
- **K** Trigonometric equations in quadratic form

Right Angle Trig - Review

Geogebra Sketch to Explore

m∠BCA = 37.02°
Hypotenuse = 5.00 cm
Opposite = 3.01 cm
Adjacent = 4.00 cm

Hide Sin $\frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{3.01 \text{ cm}}{5.00 \text{ cm}} = 0.60 = \text{Sin}(37.02°)$

Hide Cosine $\frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{4.00 \text{ cm}}{5.00 \text{ cm}} = 0.80 = \text{Cos}(37.02°)$

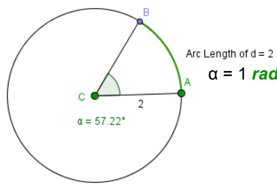
Hide Tangent $\frac{\text{Opposite}}{\text{Adjacent}} = \frac{3.01 \text{ cm}}{4.00 \text{ cm}} = 0.75 = \text{Tan}(37.02°)$

Mnemonic SOH CAH TOA
Some Old Hippie Caught Another Hippie Tripping on Acid

A RADIAN MEASURE

Lots of theories about where 360 degrees came from. The Babylonians has a base 60 number system that may have contributed. One version of a Mayan calendar used 20 cycles of 18 days (360) plus 5 unlucky days! The Persian calendar used 360 days for a year. Note the connection to hours, minutes and seconds - time was initially measured based on astronomical cycles that were assumed to be circular.

Roger Cotes is generally credited with defining a new unit of measure. The ***radian***. It was natural to measure an angle by the length of the arc that it subtends. But that length would be different for different size circles. Unless, of course, you measure the length in "numbers of radii of the given circle". Thus the name "radians".



$\alpha = 1$ ***radian***

So how do we convert between the systems?

How many radians (radii) are there in a complete circle?

Distance around a circle is $2\pi r$ so:

360 degrees = 2π radians
or 180 degrees = π radians

Try a couple:

Convert 45^o to radians in terms of π.

$45^o = (45 \times \frac{\pi}{180})$ radians *or* $180^o = \pi$ radians

$= \frac{\pi}{4}$ radians $\therefore\ (\frac{180}{4})^o = \frac{\pi}{4}$ radians

$\therefore\ 45^o = \frac{\pi}{4}$ radians

Convert $\frac{5\pi}{6}$ to degrees.

$\frac{5\pi}{6}$

$= (\frac{5\pi}{6} \times \frac{180}{\pi})^o$

$= 150^o$

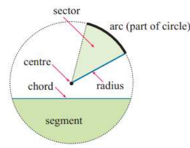
RightAngleTrigReview.PDF

Handout Review set
8A #1-4 odd cols, 5 all

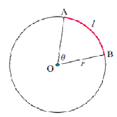
Go through Stat/Prob test

B ARC LENGTH AND SECTOR AREA

First, some vocabulary



ARC LENGTH



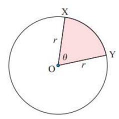
In the diagram, the **arc length** AD is l. θ is measured in **radians**.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\theta}{2\pi}$$

$$\therefore \quad \frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$l = \theta r$$

AREA OF SECTOR



In the diagram, the area of minor sector XOY is shaded. θ is measured in **radians**.

$$\frac{\text{area of minor sector XOY}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

$$\therefore \quad \frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$A = \tfrac{1}{2}\theta r^2$$

If θ is in **degrees**, $l = \frac{\theta}{360} \times 2\pi r$ and $A = \frac{\theta}{360} \times \pi r^2$.

A sector has radius 12 cm and angle 3 radians. Use radians to find its:
a arc length **b** area

a arc length $= \theta r$
$= 3 \times 12$
$= 36$ cm

b area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 3 \times 12^2$
$= 216 \text{ cm}^2$

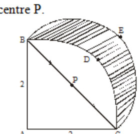
A sector has radius 8.2 cm and arc length 13.3 cm.
Find the area of this sector.

$l = \theta r$ {θ in radians}

$\therefore \quad \theta = \frac{l}{r} = \frac{13.3}{8.2}$

$\therefore \quad \text{area} = \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times \frac{13.3}{8.2} \times 8.2^2$
$\approx 54.5 \text{ cm}^2$

The diagram below shows a triangle and two arcs of circles.
The triangle ABC is a right-angled isosceles triangle, with AB = AC = 2. The point P is the midpoint of [BC].
The arc BDC is part of a circle with centre A.
The arc BEC is part of a circle with centre P.



(a) Calculate the area of the segment BDCP.
(b) Calculate the area of the shaded region BECD.

(Total 6 marks)

(a) area of sector ABDC = $\frac{1}{4}\pi(2)^2 = \pi$ (A1)
area of segment BDCP = π – area of ΔABC (M1)
$= \pi - 2$ (A1) (C3)

(b) BP = $\sqrt{2}$ (A1)
area of semicircle of radius BP = $\frac{1}{2}\pi(\sqrt{2})^2 = \pi$ (A1)
area of shaded region = $\pi - (\pi - 2) = 2$ (A1)
(C3)

[6]

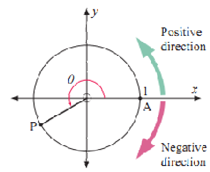
8B #1-4, 7-12 Arc length and sector area

Present 7-10,12,QB 3, 4

C THE UNIT CIRCLE AND THE BASIC TRIGONOMETRIC RATIOS

A circle of radius one is one of the most fundamental geometric shapes. It is a very effective visual reference for many ideas involving trigonometry.

Unit Circle demo



From this we see some important principles:

$\cos\theta$ is the x-coordinate of P.
$\sin\theta$ is the y-coordinate of P.

These are more general definitions since they include negative values and angles greater than 90 deg.

$-1 \leqslant \cos\theta \leqslant 1$ and $-1 \leqslant \sin\theta \leqslant 1$ for all θ.

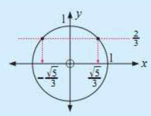
$\cos^2\theta + \sin^2\theta = 1.$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}.$ This is a more general definition of tangent.

Try some applications:

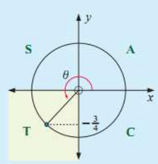
Find the possible values of $\cos\theta$ for $\sin\theta = \frac{2}{3}$. Illustrate your answers.

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \cos^2\theta + \left(\tfrac{2}{3}\right)^2 &= 1 \\ \therefore\ \cos^2\theta &= \tfrac{5}{9} \\ \therefore\ \cos\theta &= \pm\tfrac{\sqrt{5}}{3} \end{aligned}$$



Notice that trig functions are ***not one-to-one***! For example, the sin or cosine of an angle is a single value, but there are ***two*** angles between 0 and 2π that have a particular sin or cosine!

If $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\theta$ and $\tan\theta$ without using a calculator.

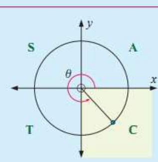


$$\begin{aligned} \text{Now}\quad \cos^2\theta + \sin^2\theta &= 1 \\ \therefore\ \cos^2\theta + \tfrac{9}{16} &= 1 \\ \therefore\ \cos^2\theta &= \tfrac{7}{16} \\ \therefore\ \cos\theta &= \pm\tfrac{\sqrt{7}}{4} \end{aligned}$$

But $\pi < \theta < \frac{3\pi}{2}$, so θ is a quadrant 3 angle
$\therefore$ $\cos\theta$ is negative.
$\therefore$ $\cos\theta = -\frac{\sqrt{7}}{4}$

and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\frac{3}{4}}{-\frac{\sqrt{7}}{4}} = \dfrac{3}{\sqrt{7}}$

If $\tan\theta = -2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\theta$ and $\cos\theta$.



$$\begin{aligned} \frac{\sin\theta}{\cos\theta} &= -2 \\ \therefore\ \sin\theta &= -2\cos\theta \\ \text{Now}\quad \sin^2\theta + \cos^2\theta &= 1 \\ \therefore\ 4\cos^2\theta + \cos^2\theta &= 1 \\ \therefore\ 5\cos^2\theta &= 1 \\ \therefore\ \cos\theta &= \pm\tfrac{1}{\sqrt{5}} \end{aligned}$$

But $\frac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, so θ is a quadrant 4 angle.

$\therefore$ $\cos\theta$ is positive and $\sin\theta$ is negative.

$\therefore$ $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$.

8C.1	#1b,2a,3,4,5&6*,7bdf,8b,9
8C.2	#1cd,2cd,3bd,4cd,5bd
QB	3,10,13,20

Review of some important results from last time:

$\cos\theta$ is the x-coordinate of P.
$\sin\theta$ is the y-coordinate of P.

These are more general definitions since they include negative values and angles greater than 90 deg.

$-1 \leqslant \cos\theta \leqslant 1$ and $-1 \leqslant \sin\theta \leqslant 1$ for all θ.

$\cos^2\theta + \sin^2\theta = 1.$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}.$ This is a more general definition of tangent.

Since there are 2π radians in a full revolution, if we add any integer multiple of 2π to θ then the position of P on the unit circle is unchanged.

So, for all $k \in \mathbb{Z}$ and angles θ, $\cos(\theta + 2k\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$.

Some results that you may have developed. **Don't** memorize. **Do** understand:

sin(180 - θ) = sinθ	cos(180 - θ) = -cosθ
sin(90 - θ) = cosθ	cos(90 - θ) = sinθ
sin(-θ) = - sinθ	cos(-θ) = cosθ

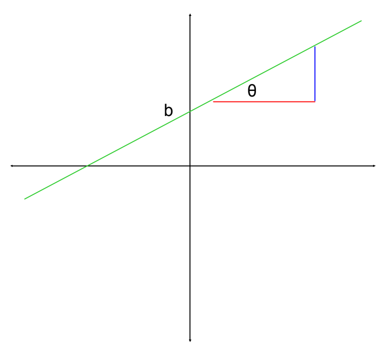
The most common angles are multiples of 30 or 45 degrees.
These correspond to multiples of π/6 and π/4 radians.

Because they are related to the common 30-60-90 and 45-45-90 triangles, the values of the sin and cos functions for these angles should be memorized.

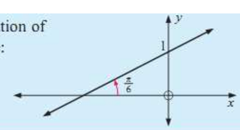
You need to be able to quickly construct a unit circle with exact coordinates, degree and radian measures of all angles that are multiples of 30 or 45 degrees.

- If θ is a **multiple of** $\frac{\pi}{2}$, the coordinates of the points on the unit circle involve 0 and ± 1.
- If θ is a **multiple of** $\frac{\pi}{4}$, but not a multiple of $\frac{\pi}{2}$, the coordinates involve $\pm\frac{1}{\sqrt{2}}$.
- If θ is a **multiple of** $\frac{\pi}{6}$, but not a multiple of $\frac{\pi}{2}$, the coordinates involve $\pm\frac{1}{2}$ and $\pm\frac{\sqrt{3}}{2}$.

D THE EQUATION OF A STRAIGHT LINE



Find the equation of the given line:

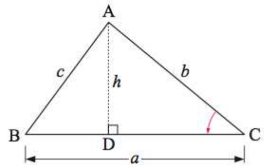


The line has gradient $m = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$
and its y-intercept is 1.
$\therefore$ the line has equation $y = \frac{1}{\sqrt{3}}x + 1$.

8C.3: #1-5 & 7 (Multiples of pi/4 & pi/6)
8D: #1&2 (Straight line & tangents)
QB: 27,33,37,45 (IB Practice)

A AREAS OF TRIANGLES

We can use trig to calculate the area of a triangle. First, consider the acute triangle below:



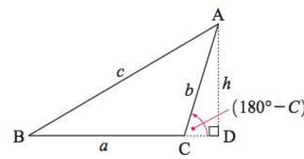
We know area to be A = ½*ha*

But notice that sin*C* = *h*/*b* so *h* = *b* sin*C*
Thus,

$A = \frac{1}{2}\, a\, b \sin C$

We call C the ***included angle*** between *a* and *b*.

What about obtuse triangles?

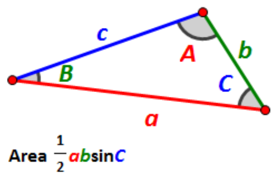


Again A = ½*ha*

But now *C* is obtuse and our right triangle has an angle of 180° - *C*.

We've already seen that sin(180° - *C*) = sin *C* = *h*/*b* so it's still true that *h* = *b* sin*C*
And, again,

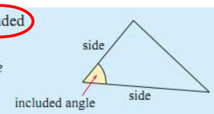
$A = \frac{1}{2}\, a\, b \sin C$



$A = \frac{1}{2}ab\sin C.$

Given the lengths of two sides of a triangle and the included angle between them, the area of the triangle is

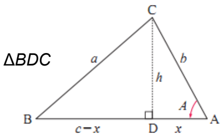
a half of the product of two sides and the sine of the included angle.



9A #1-11 Areas of Triangles

B THE COSINE RULE

Consider the triangle at right:



Note that h can be computed from ΔADC or from ΔBDC

Thus $b^2 - x^2 = a^2 - (c - x)^2$ (Pythagorus)

Let's simplify:

$b^2 - x^2 = a^2 - (c^2 - 2cx + x^2)$
$b^2 - x^2 + (c^2 - 2cx + x^2) = a^2$
$b^2 - x^2 + c^2 - 2cx + x^2 = a^2$
$b^2 + c^2 - 2cx = a^2$

But trig tells us that $x = b\cos A$ (Definition of cos)

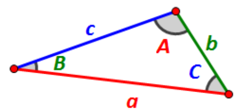
Substitute to get $b^2 + c^2 - 2bc\cos A = a^2$ The ***cosine rule***

The Law of Cosines (Cosine Rule)

For a triangle with sides of length *a*, *b*, & *c* whose opposite angles are given by *A*, *B*, & *C* respectively, the following relationships hold:

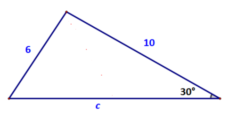
Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Notice that the angle is the ***included angle*** between the two known sides. If the ***non-included angle*** is given, the situation is ambiguous as the resulting equation will be quadratic, with two potential solutions.

Consider:



$6^2 = 10^2 + c^2 - 2(10)c\cos 30^\circ$

$36 = 100 + c^2 - 20c\left(\frac{\sqrt{3}}{2}\right)$

$0 = c^2 - 10\sqrt{3}c + 64$

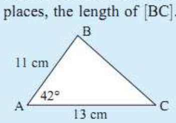
So $c = 11.98$ or 5.34

The good news is that using the cosine rule will make it clear that there are multiple solutions because the equation will tell you that!

Some rearranging gives some other useful forms:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{c^2 + a^2 - b^2}{2ca} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Find, correct to 2 decimal places, the length of [BC].



By the cosine rule:

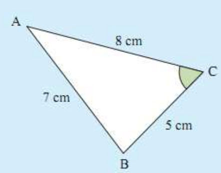
$\mathrm{BC}^2 = 11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42^o$

$\therefore$ $\mathrm{BC} \approx \sqrt{(11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42^o)}$

$\therefore$ $\mathrm{BC} \approx 8.801$

$\therefore$ [BC] is 8.80 cm in length.

In triangle ABC, if AB = 7 cm, BC = 5 cm and CA = 8 cm, find the measure of angle BCA.



By the cosine rule:

$\cos C = \dfrac{(5^2 + 8^2 - 7^2)}{(2 \times 5 \times 8)}$

$\therefore$ $C = \cos^{-1}\left(\dfrac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8}\right)$

$\therefore$ $C = 60^o$

So, angle BCA measures 60^o.

9B: #1-7 (Cosine Rule)

C THE SINE RULE

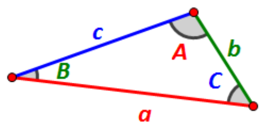
We have shown that the area of a triangle can be given as:
$\frac{1}{2}ab \cdot \sin C = \frac{1}{2}ac \cdot \sin B = \frac{1}{2}ab \cdot \sin C$

By multiplying all three expressions by 2 and dividing them all by the product *abc* we get the ***sine rule.***

The Law of Sines (Sine Rule)

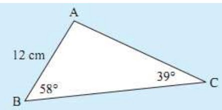
For a triangle with sides of length *a*, *b*, & *c* whose opposite angles are given by *A*, *B*, & *C* respectively, the following relationships hold:

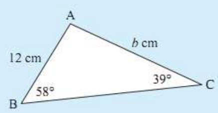
Law of Sines $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



You can use the sine rule to find sides or angles. Some cases are straightforward.

Find the length of [AC] correct to two decimal places.





Using the sine rule, $\frac{b}{\sin 58°} = \frac{12}{\sin 39°}$

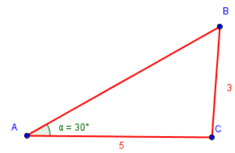
$\therefore \quad b = \frac{12 \times \sin 58°}{\sin 39°}$

$\therefore \quad b \approx 16.170\,74$

$\therefore$ [AC] is about 16.17 cm long.

Now draw a triangle with $a = 3$, $b = 5$ and $\angle A = 30°$ *and solve the triangle using the sine rule.*

Did you draw this?



$$\frac{\sin B}{5} = \frac{\sin 30°}{3}$$

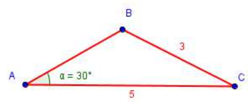
$$\sin B = 5\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{5}{6}$$

$$B = \sin^{-1}\left(\frac{5}{6}\right) \approx 56.44°$$

so $C = 180° - 30° - 56.44° \approx 93.56°$

and $\frac{\sin 30°}{3} = \frac{1}{6} = \frac{\sin 93.56°}{c}$ or $c = 6 \sin 93.56° \approx 5.988$

Did anyone draw this?



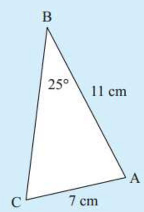
The Law of Sines gives you exactly the same result! How can this be?

The answer lies in the fact that there are ***two*** angles < 180° that have a sin of 5/6. One is 56.44 which your calculator provides. But the other is 180° - 56.44° = 123.56°. So angle *B* might also be 123.56°! Now notice the given angle, 30°. If we add it to 123.56° we end up less than 180° so there is still 180° - (123.56° + 30°) = 26.47° left for angle *C*. Thus we have a ***second solution***!

The ***Ambiguous Case*** of the Law of Sines

When given two sides and a ***non-included*** angle, it is possible two have one, two or no solutions to the triangle!
You must explore both solutions to $\sin^{-1}$. Using the cosine rule will help!

Find the measure of angle C in triangle ABC if AC = 7 cm, AB = 11 cm, and angle B measures 25^o.



$$\frac{\sin C}{c} = \frac{\sin B}{b} \quad \{\text{by the sine rule}\}$$

$$\therefore \quad \frac{\sin C}{11} = \frac{\sin 25^o}{7}$$

$$\therefore \quad \sin C = \frac{11 \times \sin 25^o}{7}$$

$$\therefore \quad C = \sin^{-1}\left(\frac{11 \times \sin 25^o}{7}\right) \text{ or its supplement}$$

$$\therefore \quad C \approx 41.6^o \text{ or } 180^o - 41.6^o$$

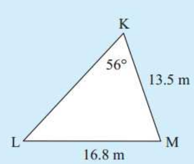
$\{$as C may be obtuse$\}$

$$\therefore \quad C \approx 41.6^o \text{ or } 138.4^o$$

$\therefore$ C measures 41.6^o if angle C is acute, or 138.4^o if angle C is obtuse.

In this case there is insufficient information to determine the actual shape of the triangle.

Find the measure of angle L in triangle KLM given that angle LKM measures 56^o, LM = 16.8 m, and KM = 13.5 m.



$$\frac{\sin L}{13.5} = \frac{\sin 56^o}{16.8} \quad \{\text{by the sine rule}\}$$

$$\therefore \quad \sin L = \frac{13.5 \times \sin 56^o}{16.8}$$

$$\therefore \quad L = \sin^{-1}\left(\frac{13.5 \times \sin 56^o}{16.8}\right) \text{ or its supplement}$$

$$\therefore \quad L \approx 41.8^o \text{ or } 180^o - 41.8^o$$

$$\therefore \quad L \approx 41.8^o \text{ or } 138.2^o$$

We reject $L \approx 138.2^o$, since $138.2^o + 56^o > 180^o$ which is impossible.

$\therefore$ $L \approx 41.8^o$.

D USING THE SINE AND COSINE RULES

In problems involving ***solving*** triangles, you have three options:

> Try to find right triangles and use definitions
> Use the cosine rule
> Use the sine rule

Often more than one approach can work. Choose the one that is easiest.

Use the **cosine rule** when given:

- three sides
- two sides and an included angle.

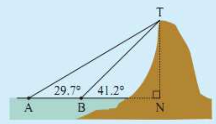
Use the **sine rule** when given:

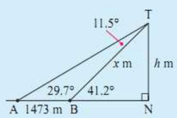
- one side and two angles
- two sides and a non-included angle, but beware of the *ambiguous case* which can occur when the smaller of the two given sides is opposite the given angle.

The angles of elevation to the top of a mountain are measured from two beacons A and B at sea.

These angles are as shown on the diagram.

If the beacons are 1473 m apart, how high is the mountain?





$A\hat{T}B = 41.2^\circ - 29.7^\circ$ {exterior angle of Δ}
$= 11.5^\circ$

We find x in ΔABT using the sine rule:

$$\frac{x}{\sin 29.7^\circ} = \frac{1473}{\sin 11.5^\circ}$$

$$\therefore \quad x = \frac{1473}{\sin 11.5^\circ} \times \sin 29.7^\circ$$

$$\approx 3660.62$$

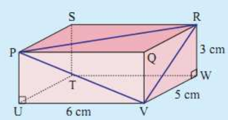
Now, in ΔBNT, $\sin 41.2^\circ = \frac{h}{x} \approx \frac{h}{3660.62}$

$\therefore \quad h \approx \sin 41.2^\circ \times 3660.62$
$\therefore \quad h \approx 2410$

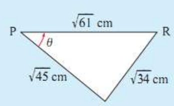
So, the mountain is about 2410 m high.

Example 9 **Self Tutor**

Find the measure of angle RPV.



In ΔRVW, $RV = \sqrt{5^2 + 3^2} = \sqrt{34}$ cm. {Pythagoras}
In ΔPUV, $PV = \sqrt{6^2 + 3^2} = \sqrt{45}$ cm. {Pythagoras}
In ΔPQR, $PR = \sqrt{6^2 + 5^2} = \sqrt{61}$ cm. {Pythagoras}



By rearrangement of the cosine rule,

$$\cos\theta = \frac{(\sqrt{61})^2 + (\sqrt{45})^2 - (\sqrt{34})^2}{2\sqrt{61}\sqrt{45}}$$

$$= \frac{61 + 45 - 34}{2\sqrt{61}\sqrt{45}}$$

$$= \frac{72}{2\sqrt{61}\sqrt{45}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{36}{\sqrt{61}\sqrt{45}}\right) \approx 46.6^\circ$$

$\therefore$ angle RPV measures about 46.6°.

9C.1: #1-2 (Sine Rule - Finding sides)
9C.2: #1-8 (Sine Rule - Ambiguous case)
9D: #2-16 even (Using sin & cos rules)
QB 29,32,39,41,43,46,49,50