Infinite Surds

Standard Level Type I

Mr. Christie

2/23/2010

The first three terms in the sequence of the infinite surd $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}$ are shown below:

$a_1 = \sqrt{1+\sqrt{1}}$

$a_2 = \sqrt{1+\sqrt{1+\sqrt{1}}}$

$a_3 = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}$

The next term in this sequence can be found by adding one to the previous term and taking the square root of the result, expressed in the formula:

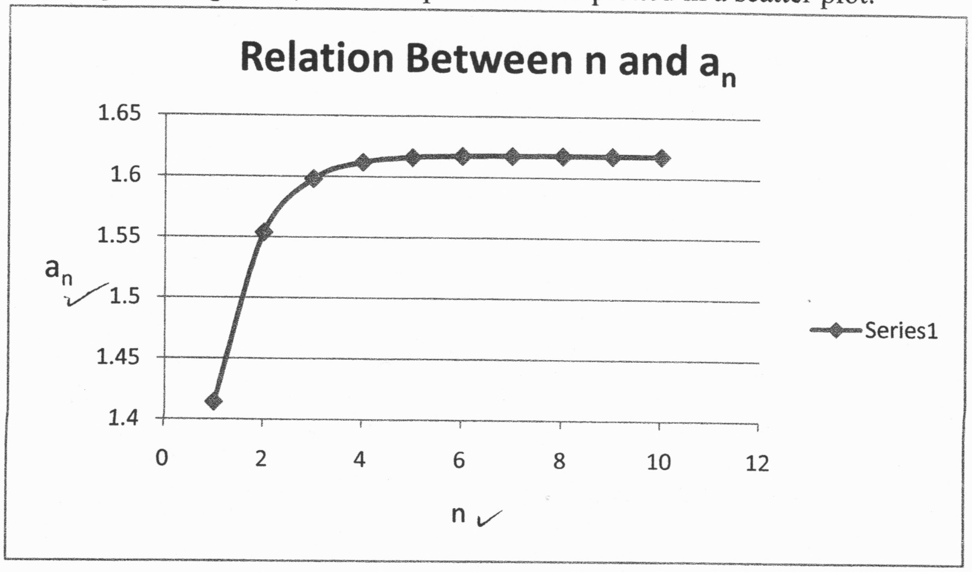
$a_{n+1} = \sqrt{1+a_n}$ ✓

The decimal values of the first ten terms of this sequence can be calculated in excel using the equation above.

n	a_n
1	1.414214
2	1.553774
3	1.598053
4	1.611848
5	1.616121
6	1.617443
7	1.617851
8	1.617978
9	1.618017
10	1.618029

✓

Then, again using excel, these ten points can be plotted in a scatter plot:



Looking at the graph I noticed that as n gets larger a_n seems to be approaching some number. I also noticed that as n gets very large, $a_n - a_{n+1}$ approaches 0. This means that if $a_n - a_{n+1} = 0$ then $a_n = a_{n+1}$. The exact value of this equation can be found as follows (assuming $a_n = a_{n+1}$):

$$a_n = \sqrt{1 + a_n}$$

$$(a_n)^2 = \left(\sqrt{1 + a_n}\right)^2$$

$$a_n^{\,2} = 1 + a_n$$

$$a_n^{\,2} - a_n - 1 = 0$$ Nice!

Then the quadratic equation can be applied

$$\frac{-(-1) + \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$\frac{1 + \sqrt{5}}{2}$$

Above is the exact value of the equation. I noticed that this value is also the Golden Ratio.

Using the same process as before I was able to determine the exact value of the infinite surd $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$ where the first term is $\sqrt{2+\sqrt{2}}$

$$a_n = \sqrt{2+a_n}$$

$$a_n{}^2 = 2 + a_n$$

$$a_n{}^2 - a_n - 2 = 0$$

$$\frac{-(-1)+\sqrt{(-1)^2-4(1)(-2)}}{2(1)}$$

$$\frac{1+\sqrt{1+8}}{2}$$

$$\frac{1+\sqrt{9}}{2}$$

$$\frac{4}{2} = 2 \checkmark$$

When solving the equation for the infinite surd with 2 it turns out that the answer is an integer. The answer found by the equation can be checked by using excel:

1	1.847759
2	1.961571
3	1.990369
4	1.997591
5	1.999398
6	1.999849
7	1.999962
8	1.999991
9	1.999998
10	1.999999
11	2
12	2

More generally, this process could be applied to any infinite surd $\sqrt{k+\sqrt{k+\sqrt{k+\sqrt{k+\cdots}}}}$ where the first term is $\sqrt{k+\sqrt{k}}$

$$a_n = \sqrt{k + a_n}$$

$$a_n{}^2 = k + a_n$$

$$a_n{}^2 - a_n - k = 0$$

$$\frac{-(-1) + \sqrt{(1)^2 - 4(1)(-k)}}{2(1)}$$

$$\frac{1 + \sqrt{1 + 4k}}{2}$$

Therefore, the general equation for an infinite surd is:

$$\frac{1 + \sqrt{1 + 4k}}{2}$$ ✓

This general equation can be tested where *k* is equal to any whole number

If k = 15 then:

$$\frac{1 + \sqrt{1 + 4(15)}}{2}$$ ✓

$$\frac{1 + \sqrt{61}}{2} = 4.405124838$$

When checked by excel

1	4.344305
2	4.398216
3	4.404341
4	4.405036
5	4.405115
6	4.405124
7	4.405125
8	4.405125
9	4.405125
10	4.405125

Verification

Both the general equation and the original formula yield the same results.

In some cases the value of the infinite surd is equal to an integer. Another equation can be generated to find the *k* value which would make the general equation equal an integer. I noticed that $1 + 4k$ could also be found by $(2a + 1)^2$ where *a* equals the term number. From there the equation can be found for *k*.

↳ Why?

$$(2a+1)^2 = 1 + 4k$$

$$(2a+1)(2a+1) = 1 + 4k$$

$$4a^2 + 4a + 1 = 1 + 4k$$

$$k = a^2 + a$$

When this equation is applied to find k, the k value found will produce an integer when plugged into the general equation.

$$2^2 + 2 = k$$

$$k = 6; a = 2$$

$$\frac{1+\sqrt{1+4(6)}}{2} = \frac{1+\sqrt{25}}{2} = \frac{6}{2} = 3$$

$$7^2 + 7 = k$$

$$k = 12; a = 7$$

$$\frac{1+\sqrt{1+4(12)}}{2} = \frac{1+\sqrt{49}}{2} = \frac{8}{2} = 4$$

$$20^2 + 20 = k$$

$$k = 420; a = 20$$

$$\frac{1+\sqrt{1+4(420)}}{2} = \frac{1+\sqrt{1681}}{2} = \frac{1+41}{2} = \frac{42}{2} = 21$$

Did you verify these using technology? How do you know that these are all values?

$$450^2 + 450 = k$$

$$k = 202{,}950; a = 450$$

$$\frac{1+\sqrt{1+4(202{,}950)}}{2} = \frac{1+\sqrt{811801}}{2} = \frac{1+901}{2} = \frac{902}{2} = 451$$

To arrive at the equation $k = a^2 + a$ where k will produce a solution which is an integer I first had to find and equation for the next term of any given sequence in the form $\sqrt{k+\sqrt{k}}$. By graphing the sequence I found that as *n* gets very large, $a_n - a_{n+1} = 0$. By assuming that this was true I was able to deduce that in this situation $a_n = a_{n+1}$. From there I was able to formulate a general equation for any given integer, *k*. When *k* was plugged into the equation $a_n = \sqrt{k + a_n}$ it produced the general equation $\frac{1+\sqrt{1+4k}}{2}$. From this general equation I noticed that some values turned out to be integers. I was then able to find an equation to find *k* values which would give an integer when plugged into the general equation. I noticed that $(2a + 1)^2$, where *a* is the term number, is equal to the portion of the general equation inside the square root, $1 + 4k$. I set them equal to each other and solved for *k* producing the equation $k = a^2 + a$. When this equation is solved for *k* that *k*-value will generate an integer sum in the general equation.

The major limitation is that both the general equation and the equation to find *k* are designed to work with whole numbers. Another limiting factor is that the general equation works on sequences in which the first term is of the nature $\sqrt{k + \sqrt{k}}$. This also limits the equation for *k* because it was found assuming that the first term of any sequence was $\sqrt{k + \sqrt{k}}$. If the form of the sequence was changed then it is likely that the equation for *k* and the general equation would not be appropriate.

why?

still dont know where this is from?