EXERCISES 1.1.1

1 (a) $u_n = \frac{1}{n(n+1)}$. Let $n = 1, 2, 3, 4$, giving

the first four terms $\frac{1}{1.2}, \frac{1}{2.3}, \frac{1}{3.4}, \frac{1}{4.5}$.

That is: $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}$.

(b) $u_n = 3 + \frac{n}{3}$. Let $n = 1, 2, 3, 4$. giving

the first four terms $3+\frac{1}{3}, 3+\frac{2}{3}, 3+\frac{3}{3}, 3+\frac{4}{3}$.

That is: $\frac{10}{3}, \frac{11}{3}, 4, \frac{13}{3}$.

(c) $u_n = 5 + \frac{1}{n}$. Let $n = 1, 2, 3, 4$, giving

the first four terms $5+\frac{1}{1}, 5+\frac{1}{2}, 5+\frac{1}{3}, 5+\frac{1}{4}$.

That is: $6, 5\frac{1}{2}, 5\frac{1}{3}, 5\frac{1}{4}$.

(d) $u_n = n + 2^n$. Let $n = 1, 2, 3, 4$, giving

the first four terms $1+2^1, 2+2^2, 3+2^3, 4+2^4$.

That is: $3, 6, 11, 20$.

2 (a) $u_n = \frac{4n^2}{2n^2 - n}$. Let $n = 1, 2, 3, 4$, giving

$\frac{4(1)^2}{2(1)^2 - 1}, \frac{4(2)^2}{2(2)^2 - 2}, \frac{4(3)^2}{2(3)^2 - 3}, \frac{4(4)^2}{2(4)^2 - 4}$

The first four terms: $4, \frac{8}{3}, \frac{12}{5}, \frac{16}{7}$.

(b) $u_n = \frac{\sqrt{n}}{n}$. Let $n = 1, 2, 3, 4$, giving $\frac{\sqrt{1}}{1}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{3}, \frac{\sqrt{4}}{4}$.

The first four terms: $\underline{1, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{3}, \frac{1}{2}}$.

(c) $u_n = \frac{n}{(n+1)^2}$. Let $n = 1, 2, 3, 4$, giving

$$\frac{1}{(1+1)^2}, \frac{2}{(2+1)^2}, \frac{3}{(3+1)^2}, \frac{4}{(4+1)^2}$$

The first four terms: $\underline{\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}}$.

3 (a) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ $u_n = \frac{1}{3^{n-1}}, \ n \in \mathbb{Z}^+$.

(b) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \ldots$ $u_n = \frac{1}{(-3)^{n-1}}, \ n \in \mathbb{Z}^+$.

(c) $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \ldots$ $u_n = \frac{1}{n^2+1}, \ n \in \mathbb{Z}^+$.

4 (a) $u_n = \sqrt{2n} + 2$. Let $n = 1, 5, 10, 50$, giving

$\sqrt{2} + 2, \ \sqrt{10} + 2, \ \sqrt{20} + 2, \ \sqrt{100} + 2$.

$\underline{u_1 = 3.4142; \ u_5 = 5.1623; \ u_{10} = 6.4721; \ u_{50} = 12.}$

(b) $u_n = \frac{n}{\sqrt{n}} = \sqrt{n}$. Let $n = 1, 5, 10, 50$, giving

$\underline{u_1 = 1; \ u_5 = 2.2361; \ u_{10} = 3.1623; \ u_{50} = 7.0711.}$

(c) $u_n = \arctan(n)$. Let $n = 1, 5, 10, 50$, giving

$u_1 = \arctan(1) = \pi/4 = \underline{0.7854}$;

$u_5 = \arctan(5) = \underline{1.3734}$;

$u_{10} = \arctan(10) = \underline{1.4711}$;

$u_{50} = \arctan(50) = \underline{1.5508}$.

5 (a) $u_n = n^2 \sin\left(\frac{\pi}{2^n}\right)$. Let $n = 1, 2, 3, 4$, giving

$1^2 \sin\left(\frac{\pi}{2}\right), \quad 2^2 \sin\left(\frac{\pi}{4}\right), \quad 3^2 \sin\left(\frac{\pi}{8}\right), \quad 4^2 \sin\left(\frac{\pi}{16}\right)$

First four terms: $\underline{1, \frac{4}{\sqrt{2}}, 9\sin\left(\frac{\pi}{8}\right), 16\sin\left(\frac{\pi}{16}\right)}$.

(b) $u_n = \sqrt{n+2} - \sqrt{n}$. Let $n = 1, 2, 3, 4$, giving

$\sqrt{3} - \sqrt{1}, \quad \sqrt{4} - \sqrt{2}, \quad \sqrt{5} - \sqrt{3}, \quad \sqrt{6} - \sqrt{4}$.

First four terms: $\underline{\sqrt{3} - 1, 2 - \sqrt{2}, \sqrt{5} - \sqrt{3}, \sqrt{6} - 2}$.

(c) $u_n = \frac{n}{(n+1)!}$. Let $n = 1, 2, 3, 4$, giving

$\frac{1}{2!}, \quad \frac{2}{3!}, \quad \frac{3}{4!}, \quad \frac{4}{5!}$.

First four terms: $\underline{\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{1}{30}}$.

6 (a) $\left\{\frac{n}{2} + 1\right\}_{n=1}^{\infty} = \frac{1}{2} + 1, \frac{2}{2} + 1, \frac{3}{2} + 1, \frac{4}{2} + 1, \frac{5}{2} + 1, \ldots$

$= \underline{\frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, \ldots}$

(b) $\left\{\frac{5}{n^2}\right\}_{n=1}^{\infty} = \frac{5}{1^2}, \frac{5}{2^2}, \frac{5}{3^2}, \frac{5}{4^2}, \frac{5}{5^2}, \ldots$

$= \underline{5, \ 5/4, \ 5/9, \ 5/16, \ 1/5, \ldots}$

(c) $\left\{1-\frac{1}{2^n}\right\}_{n=1}^{\infty} = 1-\frac{1}{2}, 1-\frac{1}{2^2}, 1-\frac{1}{2^3}, 1-\frac{1}{2^4}, 1-\frac{1}{2^5}, \ldots$

$= \underline{1/2, \ 3/4, \ 7/8, \ 15/16, \ 31/32, \ldots}$

(d) $\left\{\sin\left(\frac{\pi}{n}\right)\right\}_{n=1}^{\infty}$

$= \sin(\pi), \sin\left(\frac{\pi}{2}\right), \sin\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{5}\right), \ldots$

$= 0, 1, \sqrt{3}/2, 1/\sqrt{2}, \sin\left(\frac{\pi}{5}\right), \ldots$

$= \underline{0, 1, 0.8660, 0.7071, 0.5878, \ldots}$

7 (a) $\left\{\left(1+\frac{1}{2n}\right)^{2n}\right\}_{n=1}^{\infty}$

$= \left(1+\frac{1}{2}\right)^2, \left(1+\frac{1}{4}\right)^4, \left(1+\frac{1}{6}\right)^6, \left(1+\frac{1}{8}\right)^8, \left(1+\frac{1}{10}\right)^{10}, \ldots$

$= 9/4, \ (5/4)^4, \ (7/6)^6, \ (9/8)^8, \ (11/10)^{10}, \ldots$

$= \underline{1.25, \ 2.4414, \ 2.5216, \ 2.5658, \ 2.5937, \ldots}$

(b) $\left\{\frac{\ln(n)}{n}\right\}_{n=1}^{\infty} = \frac{\ln(1)}{1}, \frac{\ln(2)}{2}, \frac{\ln(3)}{3}, \frac{\ln(4)}{4}, \frac{\ln(5)}{5}, \ldots$

$= \underline{0, \ 0.3466, \ 0.3662, \ 0.3466, \ 0.3219, \ldots}$

(c) $\left\{ n \sin\left(\frac{\pi}{n}\right) \right\}_{n=1}^{\infty}$

$= \sin(\pi),\ 2\sin\left(\frac{\pi}{2}\right),\ 3\sin\left(\frac{\pi}{3}\right),\ 4\sin\left(\frac{\pi}{4}\right),\ 5\sin\left(\frac{\pi}{5}\right), \ldots$

$= 0,\ 2,\ \frac{3\sqrt{3}}{2},\ \frac{4}{\sqrt{2}},\ 5\sin\left(\frac{\pi}{5}\right), \ldots$

$= \underline{0,\ 2,\ 2.5981,\ 2.8284,\ 2.9389, \ldots}$

(8) $u_{n+1} = 2\left(\sqrt{u_n} + 1\right)$, $u_1 = 1$. Let $n = 1, 2, 3$.

$\Rightarrow u_2 = 2\left(\sqrt{1} + 1\right) = 4;\ u_3 = 2\left(\sqrt{4} + 1\right) = 6;\ u_4 = 2\left(\sqrt{6} + 1\right)$

First four terms: $\underline{1,\ 4,\ 6,\ 2\left(\sqrt{6} + 1\right)}$.

(9) $u_{n+2} = \sqrt{u_n \cdot u_{n+1}}$, $u_1 = 2$ and $u_2 = 4$.

Let $n = 1, 2, 3$:

$u_3 = \sqrt{u_1 \cdot u_2} = \sqrt{2.4} = 2\sqrt{2} = 2^{3/2}$;

$u_4 = \sqrt{u_2 \cdot u_3} = \sqrt{4 \cdot 2^{3/2}} = \sqrt{2^{7/2}} = 2^{7/4}$;

$u_5 = \sqrt{u_3 \cdot u_4} = \sqrt{2^{3/2} \cdot 2^{7/4}} = \sqrt{2^{13/4}} = 2^{13/8}$.

First five terms: $\underline{2,\ 4,\ 2^{3/2},\ 2^{7/4},\ 2^{13/8}}$.

(10) $u_n = u_{n-1} + u_{n-2}$, $n \geq 3$, $u_1 = 1$, $u_2 = 1$.

$u_3 = u_2 + u_1 = 2$; $\quad u_4 = u_3 + u_2 = 2 + 1 = 3$;

$u_5 = u_4 + u_3 = 3 + 2 = 5$; $\quad u_6 = u_5 + u_4 = 5 + 3 = 8$.

First six terms: 1, 1, 2, 3, 5, 8. [Fibonacci sequence]

(11) $\{u_n\}_{n=0}^{\infty}$ where $u_{n+1} = \frac{1}{2}(u_n + u_{n-1}),\ n \geq 1$

$u_0 = b, \quad u_1 = a.$

$$
\begin{aligned}
u_n - u_{n-1} &= \frac{1}{2}\left(u_{n-1} + u_{n-2} - (u_{n-2} + u_{n-3})\right) \\
&= \frac{1}{2}\left(u_{n-1} - u_{n-3}\right) \\
&= \frac{1}{2}\left[\frac{1}{2}(u_{n-2} + u_{n-3}) - u_{n-3}\right] \\
&= \left(\frac{1}{2}\right)^2 \left[u_{n-2} - u_{n-3}\right] \quad \text{—— (1)} \\
&= \left(\frac{1}{2}\right)^2 \left[\frac{1}{2}(u_{n-3} + u_{n-4}) - u_{n-3}\right] \\
&= \left(\frac{1}{2}\right)^3 \left[u_{n-4} - u_{n-3}\right] \\
&= -\left(\frac{1}{2}\right)^3 \left[u_{n-3} - u_{n-4}\right] \quad \text{—— (2)}
\end{aligned}
$$

Setting $u_{n-3} = u_{n-4} + u_{n-5}$ gives:

$$
\begin{aligned}
&\vdots \\
&= \left(\frac{1}{2}\right)^4 \left[u_{n-4} - u_{n-5}\right] \quad \text{—— (3)}
\end{aligned}
$$

Following the pattern of (1) → (2) → (3) gives:

$$
\begin{aligned}
u_n - u_{n-1} &= \left(-\frac{1}{2}\right)^5 \left[u_{n-5} - u_{n-6}\right] \\
&= \ \vdots \\
&= \left(-\frac{1}{2}\right)^n \left[u_{n-n} - u_{n-(n+1)}\right] \\
&= \left(-\frac{1}{2}\right)^{n-1} \left[u_{n-(n-1)} - u_{n-(n)}\right] \\
&= \left(-\frac{1}{2}\right)^{n-1} \left[u_1 - u_0\right] \\
&= \underline{\left(-\frac{1}{2}\right)^{n-1} (a-b), \quad n > 1.}
\end{aligned}
$$